Kähler geometry

How to get a metric with $U(\frac{n}{2})$ holonomy?

Def An almost complex structure on M is $J \in \mathcal{E}(\mathrm{End}\, TM)$ with $J^2 = -1$.

Rk An almost complex structure gives a $GL(n/2, \mathbb{C})$-structure

$$Q = \{\text{frames which have } e_{2k+1} = J e_{2k}\}$$

$$\text{or } \quad Q = \{ e : \mathbb{R}^n \to TM \} \qquad J_0 = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & \ddots & \end{pmatrix}$$

$\circlearrowleft J_0 \quad \circlearrowleft J$

Is this reduction compatible with the Levi-Civita connection?
(i.e. is the L-C conn. induced from a connection in the reduced bundle?)

Here's an evident necessary condition:

Def J is a complex structure if $\exists$ a connection in Q which induces a torsion-free connection in TM.

Rk For any G-str. $Q \subset P$, $\exists$ an obstruction to Q inducing a torsion-free connection: "intrinsic torsion". For J, this turns out to be Nijenhuis tensor,

$$N_J(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY]$$

Prop If J is a complex structure, then $\exists$ a complex atlas on M which induces J. [Newlander-Nirenberg]

Def Given (M,J,g) almost complex Riemannian mfd, say g is Hermitian if

$$\langle JX, JY \rangle = \langle X, Y \rangle.$$

Rk A Hermitian metric gives a reduction of the bundle of frames from $GL(\frac{n}{2}, \mathbb{C})$ to $U(\frac{n}{2})$.

But again we can ask whether this reduction is compatible with Levi-Civita.

Def Given (M,J,g) Hermitian, Hermitian form $\omega \in \mathcal{E}(\Lambda^2 T^*M)$ is

$$\omega(X,Y) = g(JX,Y)$$

Prop TFAE:
- J is a complex structure and $d\omega = 0$
- $\nabla J = 0$
- $\nabla \omega = 0$
- $\mathrm{Hol}(g) \subset U(m)$ and J is assoc. to the corresp $U(m)$ structure.

The local structure of a Kähler metric is <u>much</u> simpler than for a general Riem metric...

Can always locally write $$g(\partial_i, \partial_{\bar{j}}) = \partial_i \partial_{\bar{j}} K$$

for some real-valued <u>function</u> K.