

1 Why study complex geometry?

The main goal of this course is to understand the natural *differential geometry* of *complex manifolds*: Kähler geometry.

Why should we care about this? Several possible answers.

For differential geometers: A general Riemannian metric is a difficult beast. We'd like to look for a class of metrics which is big enough to contain a lot of interesting examples, but somehow better behaved than the most general case. Kähler metrics provide such a class. Essentially all of the standard formulas of Riemannian geometry simplify when restricted to Kähler metrics. So, for instance, if you are looking for a compact Ricci-flat manifold, your job is much easier if you look at the Kähler case: existence theorems and deformation theory are both well understood. Another way of making the same point is to say that *special holonomy* manifolds are easier than general ones, and Kähler manifolds are the easiest example of this class (their holonomy is controlled by a *unitary* group rather than orthogonal).

For algebraic geometers: If you are interested in projective algebraic varieties X over \mathbb{C} , then you should be interested in Kähler metrics: every such X has one, and it leads to information about the variety that is otherwise hard to get at. One “easy” example: the odd Betti numbers of X (b_1, b_3, \dots) are all *even* (this generalizes the fact that a Riemann surface of genus g has $b_1 = 2g - 2$). In fact, we will discover a rich structure in the cohomology of a Kähler manifold.

For symplectic geometers: A Kähler metric is a particularly nice case of a symplectic structure.

For physicists: Riemannian metrics arise very often in quantum field theory, as the target spaces of sigma models. If we consider *supersymmetric* sigma models in four dimensions (or their dimensional reduction to lower dimensions) then we naturally encounter Kähler geometry.

2 Topics

1. **Linear algebra of complex vector spaces.** This is a little subtler than it first sounds — one has to understand clearly the relationship between a real vector space V and its complexification, particularly in the case when V already carries a complex structure. We will work this out and its consequences for tensor algebras, particularly the exterior algebra.
2. **Complex manifolds.** These are our basic objects of study. A complex manifold can be defined in two ways: either as a space locally modeled on an open subset of \mathbb{C}^n with holomorphic transition functions, or as a real C^∞ manifold of dimension $2n$ equipped with a *complex structure*. In general a single real manifold may admit various inequivalent complex structures. We will describe various families of examples.
3. **Differential calculus on (almost) complex manifolds.** On any C^∞ manifold one has the ((super)commutative) differential graded algebra of C^∞ differential forms, graded by degree, equipped with the exterior derivative d with $d^2 = 0$. On a complex

manifold one can refine this to include a second grading, and there is a natural decomposition $d = \partial + \bar{\partial}$ with $\partial^2 = 0$, $\bar{\partial}^2 = 0$, $\partial\bar{\partial} = -\bar{\partial}\partial$. The operators ∂ and $\bar{\partial}$ are fundamental for much of what follows.

4. **Newlander-Nirenberg theorem.** A useful way of thinking about complex structures is given by the Newlander-Nirenberg theorem, which says that a complex structure is equivalent to an almost complex structure obeying an additional condition of *integrability*. We will state this theorem and prove the easy direction (a complex structure induces an integrable almost complex structure) but probably not the hard direction.
5. **Dolbeault cohomology.** The finer decomposition of the algebra of C^∞ differential forms just mentioned corresponds to a finer decomposition of the de Rham cohomology, $H^n(X) = \bigoplus_{p+q=n} H^{p,q}(X)$. Importantly, the spaces $H^{p,q}(X)$ can also be defined using only holomorphic objects, not C^∞ ones. This purely holomorphic description is best given in the language of sheaves and sheaf cohomology, so we will take a little detour to learn the basic facts about these, which is well worth doing in any case.
6. **Holomorphic vector bundles.** On a complex manifold we have a natural notion of holomorphic vector bundle. Much like complex manifolds, holomorphic vector bundles can be defined in two ways. One way is to say they are vector bundles for which the transition functions are holomorphic. The other is to say they are vector bundles equipped with a $\bar{\partial}$ -operator obeying an “integrability” condition. The equivalence between the two is a sort of linearized version of the Newlander-Nirenberg theorem. Just as the notion of complex manifold is finer than that of C^∞ manifold, the notion of holomorphic vector bundle is finer than that of C^∞ vector bundle. Holomorphic vector bundles occur often in nature (one canonical example is the tangent bundle of a complex manifold). We will briefly discuss the classification of such bundles in simple situations, particularly over $\mathbb{C}\mathbb{P}^1$ (Grothendieck’s theorem).
7. **Holomorphic line bundles and divisors.** Holomorphic vector bundles of rank 1, also called holomorphic line bundles, play a particularly important role in complex geometry, especially in its algebraic applications. We will discuss them relatively briefly. In particular we will introduce the “Picard group” $Pic(X)$ consisting of all holomorphic line bundles on X up to isomorphism. By considering the zeroes and poles of a meromorphic section, one also discovers that line bundles are closely related to *divisors*, certain codimension-1 subsets of X given locally as zero sets of holomorphic functions.
8. **Hermitian metrics and Hermitian bundles.** On a complex manifold, a coarse notion of “metric compatible with a complex structure” is that of a *Hermitian metric* on the tangent bundle. We will describe this notion and its analogue for a general vector bundle.
9. **Kähler manifolds.** A Kähler manifold is a complex manifold equipped with a particularly congenial Hermitian metric. There are several equivalent ways of understanding what a Kähler manifold is. The most differential-geometric approach is to say it is a metric in which around every point there is a system of Riemann normal coordinates which are also holomorphic coordinates. Another approach, more adapted to the needs

e.g. of mirror symmetry, is to say it is a metric induced by a symplectic structure and a complex structure which are compatible in a precise sense.

10. **Fubini-Study metric.** Perhaps the most famous Kähler metric is the “Fubini-Study metric” on $\mathbb{C}\mathbb{P}^n$. Its importance derives largely from the fact that it induces a Kähler metric on any compact complex submanifold of $\mathbb{C}\mathbb{P}^n$, which implies that projective varieties are Kähler.
11. **Hodge theory.** For C^∞ Riemannian manifolds there is an especially important differential operator, the Laplacian Δ acting on C^∞ forms. For compact manifolds the kernel of Δ , the space of *harmonic forms*, consists of privileged representatives for the de Rham cohomology. For compact Kähler manifolds there is a similar story for the Dolbeault cohomology. The proof uses essentially the same analytic tools as in the C^∞ case; we will give a detailed sketch of the proof but probably not fill in all the details. One thing we learn immediately from this is that the cohomology of a compact Kähler manifold carries an intricate linear-algebraic structure known as a *Hodge structure*, which encapsulates a lot of information about the manifold. In particular it says that the Betti numbers b^n can be refined to *Hodge numbers* $h^{p,q}$, with $b^n = \sum_{p+q=n} h^{p,q}$. Moreover one has $H^{p,q}(X) \simeq \overline{H^{q,p}(X)}$, so $h^{p,q} = h^{q,p}$ from which the even-ness of Betti numbers mentioned in the introduction follows.
12. **Jacobians.** The set of all holomorphic line bundles on a complex manifold X forms an interesting group $Pic(X)$. In general this group is difficult to understand, but when X is Kähler we can use Hodge theory to show that it can be elegantly described as a disjoint union of complex tori.
13. **Lefschetz $sl(2)$.** The space of harmonic forms on X (which we now know is isomorphic to the cohomology) carries a natural action of the Lie algebra $sl(2)$, generated by three operators E, F, H : H is the total degree operator (shifted by the dimension of X), E is the operation of wedging with the Kähler form, F is the operation of contracting with the Kähler form. This gives a lot of information about the cohomology of X . By the way, the appearance of $sl(2)$ here might seem a bit mysterious; why should $sl(2)$, an algebra of 2×2 matrices, have anything to do with the manifold X ? It may be clarified (for physicists anyway!) by the observation that $sl(2) \simeq so(2, 1)$ and $SO(2, 1)$ would arise as the R-symmetry group in dimensional reduction from $\mathbb{R}^{3,1}$ to $\mathbb{R}^{1,0}$, so it might naturally appear in a 4-dimensional supersymmetric theory. We will make a few remarks in this direction.
14. **Lefschetz (1,1) Theorem.** There is a canonically defined map $Pic(X) \rightarrow H^{1,1}(X)$ which we will prove is *surjective* on the integral classes $H^{1,1}(X, \mathbb{Z}) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. This is the essential step in proving the “Hodge conjecture for curve classes,” as we will explain.
15. **Chern classes.** Some of the discrete topological information specifying a complex vector bundle is nicely captured by cohomological invariants known as the *Chern classes*. An especially important case is the first Chern class of a complex line bundle, which

already appeared as the map $Pic(X) \rightarrow H^{1,1}(X)$ above. We will describe the basic properties of the Chern classes.

16. **Hirzebruch-Riemann-Roch formula.** The sheaf cohomology groups $H^p(E)$ of a holomorphic vector bundle over a compact complex manifold X are somewhat delicate objects: their dimensions are not easy to calculate in general. However, there is a rather elegant formula for the alternating *sum* of the dimensions, called the “holomorphic Euler characteristic” $\chi(E)$, which only depends on the Chern classes of E (in particular it only depends on the C^∞ structure of E , not the holomorphic structure!) This formula is a special case of the celebrated Atiyah-Singer index theorem and well worth knowing. We should definitely state the formula and some of its applications, and ideally we will give at least a sketch of a proof when X is Kähler (by now there are several different proofs using quite different technologies).
17. **GAGA.** Those whose background is in algebraic geometry may wonder how the *holomorphic* point of view (where we allow arbitrary holomorphic functions in all our constructions), which we take in most of the course, relates to a purely *algebraic* point of view (where we only allow algebraic functions, not transcendental ones). The answer, fortunately, is that in many contexts the two points of view agree. This is known generally as the “GAGA principle” after Serre’s paper “Geometrie Algebrique et Geometrie Analytique.” Also relevant here is Chow’s Theorem which says that every compact complex submanifold of $\mathbb{C}\mathbb{P}^n$ is a projective variety.
18. **Kodaira/Serre vanishing.** There are some special cases where one can understand sheaf cohomology a bit better: when L satisfies an appropriate “positivity” condition, one can show for large enough m that all the higher cohomology groups $H^i(E \otimes L^m)$ are actually *trivial*. In that case the Hirzebruch-Riemann-Roch formula gives a formula for the dimension of $H^0(E \otimes L^m)$, i.e. the number of sections of $E \otimes L^m$. Such “vanishing theorems” have many applications; we could try to sketch some.
19. **Kodaira embedding.** Given a compact Kähler manifold X we may ask whether it can be embedded into $\mathbb{C}\mathbb{P}^n$ for some n . (As mentioned, by Chow’s Theorem this is the same as asking whether X lives naturally in the world of algebraic geometry.) The answer is no in general, but sometimes yes: according to Kodaira’s embedding theorem, if L is a positive line bundle on X , then for large enough m $L^{\otimes m}$ has “a lot of sections”, and one can use those sections to map X into projective space. It is often possible to show very easily that such an L exists (e.g. in the case of simply connected Calabi-Yau manifolds, for which see below.)
20. **Yau’s Theorem.** In general it is difficult to find compact manifolds for which the Ricci curvature vanishes (apart from “boring” examples like flat spaces). Remarkably there exists a large class of Kähler manifolds with this property, which are called *Calabi-Yau manifolds* since their existence was established by Yau’s proof of Calabi’s conjecture. We will certainly not prove this theorem, but we should describe some examples of Calabi-Yau manifolds and their basic properties.

21. **(Variation of) Hodge structures.** The structure on $H^*(X)$ given by Hodge theory can be described a bit more abstractly as a *Hodge structure*. If we have a family of compact Kähler manifolds depending holomorphically on parameters, then we get a family of Hodge structures which varies in a controlled way. This structure is rigid enough that whenever you find such a family it is reasonable to ask whether it actually came from a family of compact Kähler manifolds. It can be fruitful to study such variations “abstractly,” without reference to the original family; this construction plays a crucial role in mirror symmetry, and has been exploited to great effect elsewhere as well (sadly I am not very knowledgeable on the other applications, but would love an excuse to learn about them!)
22. **Mirror symmetry.** As mentioned above, a Kähler metric is a particularly nice fusion between a symplectic structure and a complex structure. There is a bizarre-sounding, but apparently very deep, theory of “mirror symmetry” which in a certain sense exchanges the two structures. The simplest case to state is that of Calabi-Yau manifolds: given a “mirror pair” of Calabi-Yau manifolds X and Y , mirror symmetry relates the symplectic geometry of X to the complex geometry of Y , and vice versa. We can probably do no more than a brief overview of how this relation is supposed to go, and perhaps a bit about its origin in physics.
23. **Berger classification of holonomy.** The holonomy group of a “generic” Riemannian manifold of dimension n is $SO(n)$. A differential-geometric way of understanding the importance of the Kähler condition is that Kähler manifolds of complex dimension m are *special holonomy manifolds*, where the holonomy group is instead $U(m) \subset SO(2m)$. According to Berger’s classification, for an irreducible manifold there are not too many other possibilities: one can have $SU(m)$ (Calabi-Yau manifolds), $Sp(k)$ (hyperkähler manifolds), $Sp(k) \times Sp(1)/\mathbb{Z}_2$ (quaternionic-Kähler manifolds), G_2 or $Spin(7)$. Each of these types has its own distinct sort of geometry, which would be fun to describe.
24. **Hyperkähler manifolds.** A hyperkähler manifold is one with a Riemannian metric which is Kähler with respect to three different complex structures I_1, I_2, I_3 obeying the quaternionic relations $I_i I_j = \epsilon_{ijk} I_k$. Very loosely one could say that “hyperkähler is to quaternions as Kähler is to complex numbers.” Hyperkähler manifolds have particularly pleasant properties — for example, they are always Ricci-flat.
25. **Twistor spaces.** Although the definition of “hyperkähler” involves just three complex structures, in fact a hyperkähler manifold naturally carries a whole S^2 worth of complex structures. It is natural to organize them into a single complex manifold \mathcal{Z} fibered over $\mathbb{C}\mathbb{P}^1$, called the *twistor space*; in this way one manages to reduce hyperkähler geometry to complex geometry. Holomorphic objects over the twistor space then get translated to more differential-geometric objects over the original hyperkähler space.
26. **Moduli spaces of Higgs bundles.** Suppose we fix a Riemann surface C , and a complex Lie group G . Then we can consider the moduli space X of flat G -connections on C . X is “obviously” a complex manifold (or more precisely it has an open subset which is). What is much more surprising, discovered by Hitchin, is that X is actually hyperkähler.