Complex Geometry: Exercise Set 7

Exercise 1

We have seen in lecture that the Laplace operator on a Riemannian manifold is elliptic. Here we briefly consider the Laplace operator on a Lorentzian manifold.

The Laplace operator acting on functions on $\mathbb{R}^{1,1}$ is given in local coordinates by

$$\Delta = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}.$$

Naively one might imagine that its properties will be similar to those of the Laplacian on $\mathbb{R}^2$ since one can just "analytically continue $t \to it$." But:

1. Explain why Δ is not an elliptic operator.
2. Show that for any C^∞ function g on the real line, $f(x,t) = g(x+t)$ obeys $\Delta f = 0$.
3. Consider the function $f(x,t) = (x+t)^{5/2}$ (with some definite choice of branch for $x+t < 0$). Taking U to be some small patch in $\mathbb{R}^{1,1}$ intersecting the diagonal $x = -t$, show that $f \in W^s(U)$ for some s, and $\Delta f = 0$, but f is not C^∞. This is thus a counterexample to a hypothetical extension of the elliptic regularity theorem to Δ.
4. Why cannot we similarly construct a counterexample to elliptic regularity for the Laplacian on $\mathbb{R}^2$ by continuing $t \to it$ in f?

Exercise 2

Let $T : V \to W$ be a map between Banach spaces, such that $\operatorname{coker} T$ is finite-dimensional. Show that T has closed image. (Hint: the only technical devices needed are *a)* the *open mapping principle* which says that a surjective bounded linear map of Banach spaces takes open sets to open sets, and *b)* the principle that if $Z = Z_1 \oplus Z_2$ as an algebraic direct sum, and Z_1 is finite-dimensional, then $Z = Z_1 \oplus Z_2$ as a topological direct sum.)

Exercise 3

1. Let ∇_i be connections on vector bundles E_i for $i = 1, 2$. Carefully describe the associated connections on $E_1 \oplus E_2$, $E_1 \otimes E_2$, and $\operatorname{Hom}(E_1, E_2)$.
2. Let ∇_i be connections on vector bundles E_i for $i = 1, 2$. Change both connections by 1-forms $a_i \in \mathcal{A}^1(X, \operatorname{End}(E_i))$ and compute the new connections on the associated bundles $E_1 \oplus E_2$, $E_1 \otimes E_2$, and $\operatorname{Hom}(E_1, E_2)$.
3. Prove that if ∇_i are compatible with Hermitian structures on E_i then the associated connections are also compatible with natural Hermitian structures on the associated bundles.
4. Prove that if ∇_i are compatible with holomorphic structures on E_i then the associated connections are also compatible with natural holomorphic structures on the associated bundles.
5. Show that a connection ∇ on a Hermitian bundle (E, h) is Hermitian if and only if $\nabla(h) = 0$, where we think of h as a section of $(E \otimes \bar{E})^*$.

Exercise 4

Suppose E, E' are two holomorphic vector bundles over a complex manifold X. An *extension* of E by E' is a holomorphic vector bundle F which sits in an exact sequence

$$0 \to E' \to F \to E \to 0. \tag{0.1}$$

A *splitting* of the extension is a map $s : E \to F$ which is a section of the projection $p : F \to E$, i.e. $p \circ s = 1$.

1. Show that a splitting induces an isomorphism $F \simeq E \oplus E'$.
2. Show that splittings always exist *locally*, i.e. we can choose a covering of X by patches U_i and give a splitting $s_i : E \to F$ in each patch.
3. Show that if we consider the analogous notion of splitting using C^∞ bundles instead of holomorphic ones, then splittings always exist globally.
4. Given local splittings s_i, we can define a Cech cocycle $\varphi \in C^1(X, \mathrm{Hom}(E, E'))$ by $\varphi_{ij} = s_i - s_j$. Check that φ is indeed a cocycle, and that its class $[\varphi] \in H^1(X, \mathrm{Hom}(E, E'))$ is independent of our choice of local splittings.
5. Given two different extensions F_1 and F_2 of E by E', we say $F_1 \simeq F_2$ if there is a holomorphic isomorphism from F_1 to F_2 which commutes with the projections to E and the maps from E'.
   Show that $F_1 \simeq F_2$ if and only if $[\varphi_1] = [\varphi_2]$. In particular, an extension F can be split globally if and only if $[\varphi] = 0$.

Exercise 5

1. Say E is a holomorphic vector bundle over a complex manifold X. Show directly that the definition of the Atiyah class $A(E)$ does not depend on the trivialization ψ_i which we used in its construction.
2. Say E is a holomorphic vector bundle over a complex manifold X. Let $J^1(E)$ be the bundle of 1-jets of sections of E. Show that there is an exact sequence of holomorphic vector bundles
   $$0 \to \Omega^1 \otimes E \to J^1(E) \to E \to 0, \tag{0.2}$$
   i.e. $J^1(E)$ is an extension of E by $\Omega^1 \otimes E$.
3. Show that a holomorphic connection in E is a splitting of the above extension.
4. Show that the Atiyah class $A(E) \in H^1(X, \Omega^1 \otimes \mathrm{End}\, E)$ is the class representing this extension, in the sense of the previous exercise. This gives a more sophisticated way of understanding the statement that $A(E)$ is the obstruction to a holomorphic connection in E.