# Complex Geometry: Exercise Set 6

## Exercise 1

Let $M$ be a real 4-manifold and let $\sigma \in \mathcal{A}^2_{\mathbb{C}}(M)$ be closed, with $\sigma \wedge \sigma = 0$ and $\sigma \wedge \bar{\sigma}$ everywhere nonvanishing. Show that $M$ admits a unique complex structure such that $\sigma$ is a holomorphic 2-form. (Hint: What should $T^{0,1}M$ be?)

## Exercise 2

(These two should have been assigned a while ago, but better late than never.)

1. Suppose $M$ is an oriented Riemannian manifold of dimension $n$. Verify the assertion from class that $\star^2 = (-1)^{k(n-k)}$ acting on $\mathcal{A}^k(M)$.
2. If $M = X$ is complex, so $n = 2m$, show that $\star^2 = (-1)^k$ acting on $\mathcal{A}^k(X)$.

## Exercise 3

1. Verify by hand that the Kähler identities hold on $\mathbb{C}^m$.
2. Verify that the Kähler identities *do not* hold for the Hermitian metric on $\mathbb{C}^2$ with fundamental form $\omega = i dz_1 \wedge d\bar{z}_1 + i(|z_1|^2 + 1) dz_2 \wedge d\bar{z}_2$. (For example, try computing $[\partial, L]$.)

## Exercise 4

Suppose $X$ is a compact Kähler manifold, of dimension $n$.

1. Show that the Kähler form $\omega$ is harmonic.
2. Show that holomorphic top-forms on $X$ are harmonic, and harmonic $(n, 0)$-forms are holomorphic. (Note that this means the space of harmonic $(n, 0)$-forms on $X$ is actually independent of the Kähler metric we choose.)
3. Show that any two cohomologous Kähler forms $\omega$, $\omega'$ are related by $\omega = \omega' + i\partial\bar{\partial} f$ for some real function $f$.

## Exercise 5

Suppose $X$ is complex. Verify the assertion from class that if $\omega$ and $\omega'$ are the Kähler forms for some Kähler metrics on $X$, then $a\omega + b\omega'$ is too, for any $a, b > 0$.

(This implies that the space of $\omega$ corresponding to Kähler metrics forms a convex cone in $H^{1,1}(X, \mathbb{R})$, called the "Kähler cone." In fact the Kähler cone is open in $H^{1,1}(X, \mathbb{R})$; this should seem plausible since the condition of positivity is an open condition. Of course, the Kähler cone may be empty.)

## Exercise 6

A closed 2-form $\omega$ on a $C^\infty$ manifold $M$ of dimension $2m$ is called a *symplectic structure* if $\omega$ is closed and nondegenerate at every point.

1. Show that a compact symplectic manifold has $b_{2k} \geq 1$ for $0 \leq k \leq m$. (Hint: show that $\omega^m$ is not exact.)
2. Show that Kähler manifolds carry natural symplectic structures.