Complex Geometry: Exercise Set 5

Exercise 1

Directly verify two assertions from class:

1. If $M = \mathbb{R}^n$ with its usual flat metric, the Laplacian Δ acting on differential forms simply acts by $\Delta(\sum_I f_I dx_I) = \sum_I \Delta(f_I) dx_I$ where the Δ on the right is the usual Laplacian acting on functions.
2. If $X = \mathbb{C}^n$ with its usual flat metric, then $\Delta_\partial = \Delta_{\bar\partial} = \frac{1}{2}\Delta$.

Exercise 2

Prove the local version of the $\partial\bar\partial$-lemma for real $(1,1)$ forms: if $\omega \in \mathcal{A}^{1,1}(U) \cap \mathcal{A}^2(U)$ for $U \subset \mathbb{C}^n$ a polydisc, with $d\omega = 0$, then $\omega = \partial\bar\partial\gamma$ for some $\gamma \in \mathcal{A}^0(U)$. (The only analytic inputs needed are the $\bar\partial$-Poincare lemma and the ordinary Poincare lemma.)

Exercise 3

This exercise fills in one of the steps I omitted in the proof that the Levi-Civita connection on a Kähler manifold preserves the complex structure operator I.

Suppose X is a complex manifold with a Hermitian metric h. Let ∇ be the Levi-Civita connection. Define $A(X, Y, Z) = h(I(\nabla_X I)Y - (\nabla_{IX} I)Y, Z)$.

1. Show that $A(X, Y, Z) = A(Y, X, Z)$. (Hint: use Exercise 3 of set 2, the vanishing of the Nijenhuis tensor.)
2. Show that $A(X, Y, Z) = -A(X, Z, Y)$.
3. Conclude that $A = 0$ and hence that $I(\nabla_X I)Y = (\nabla_{IX} I)Y$.

Exercise 4

Suppose X is Kähler and α is a closed $(1,1)$-form which is primitive (i.e. $\Lambda(\alpha) = 0$). Show that $\Delta\alpha = 0$.

Exercise 5

1. Suppose V is a vector space with compatible inner product, complex structure and fundamental form (g, I, ω). Suppose $W \subset V$ is a subspace of dimension $2m$. Choose an orientation on W; together with g this induces a volume form vol_W. Show that $\mathrm{vol}_W/\omega^m|_W \geq \frac{1}{m!}$, with equality if and only if W is a complex subspace of V, i.e. if $IW = W$.
2. Suppose X is a Kähler manifold and Y a compact submanifold of dimension $2m$. Show that $\mathrm{vol}(Y) \geq \int_Y \frac{\omega^m}{m!}$, with equality if and only if Y is a complex submanifold of X.
3. Suppose X is a Kähler manifold for which ω is exact ($\omega = d\alpha$ for some α). Show that X has no compact complex submanifolds (in particular X is not compact).

Exercise 6

Let X be a Kähler manifold. Let $I : \mathcal{A}^{\cdot}(X) \to \mathcal{A}^{\cdot}(X)$ be the standard extension of $I : TX \to TX$. Show that $[I, d] = d^c$ and $[I, d^c] = -d$.