Complex Geometry: Exercise Set 4

Exercise 1

Show that the sheafification $\mathcal{F}^+$ of a sheaf $\mathcal{F}$ is canonically isomorphic to $\mathcal{F}$ itself.

Exercise 2

Suppose $\mathcal{F}$ is a sheaf of abelian groups over M and $\phi : M \to N$ is a continuous map. Define the *direct image* $\phi_*\mathcal{F}$ by $\phi_*\mathcal{F}(U) = \mathcal{F}(\phi^{-1}(U))$.

1. Show that $\phi_*\mathcal{F}$ is a sheaf.
   (We used a special case of this implicitly in lecture when we discussed the Cech resolution: there we had the inclusion maps $i_I : U_I \hookrightarrow M$, and we defined $C^k = \oplus_{|I|=k+1}(i_I)_*(\mathcal{F}|_{U_I})$.)
2. Suppose $M \to N$ is a covering map of degree d, and $\mathcal{F}$ is the sheaf of sections of a holomorphic vector bundle of rank n. Show that $\phi_*\mathcal{F}$ is the sheaf of sections of a holomorphic vector bundle of rank nd. (It is probably simplest to use the equivalent characterization in terms of locally free $\mathcal{O}$-modules.)
3. Say $M = \mathbb{C}$, $N = \mathbb{C}$, and $\phi(z) = z^2$. What is $\phi_*(\mathcal{O})$? (There are two obvious possibilities: either $\phi_*(\mathcal{O})$ is the sheaf of sections of a rank 2 holomorphic vector bundle, or it is something more complicated because of the ramification at $z = 0$.)

Exercise 3

You may be surprised that sheaves naturally *push forward* since we have been emphasizing the point of view that a sheaf is a kind of generalization of a vector bundle, and vector bundles naturally *pull back*.

We can define the *inverse image* of a sheaf, with a bit more difficulty. Given $\phi : M \to N$ continuous, define $\phi^{-1}\mathcal{F}$ to be the sheafification of $U \mapsto \lim_{V \supset f(U)} \mathcal{F}(V)$. (Note that if $i : S \to M$ is the inclusion of a closed subset, then $(i^{-1}\mathcal{F})(S)$ is what we defined in lecture to be $\mathcal{F}(S)$.)

1. Show by example that the sheafification is really necessary in this definition.
2. Show by example that if $\mathcal{F}$ is the sheaf of sections of a holomorphic vector bundle F, $\phi^{-1}\mathcal{F}$ is generally *not* the sheaf of sections of $\phi^*\mathcal{F}$ (unfortunately). Indeed, if $\phi : X \to Y$ and $\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$-modules, then $\phi^{-1}\mathcal{F}$ is not even a sheaf of $\mathcal{O}_X$-modules. (We could say something similar about C^∞ bundles etc, replacing $\mathcal{O}$ by the sheaf of C^∞ functions, or even more generally by any sheaf of rings.)
3. To fix this problem, when $\mathcal{F}$ is an $\mathcal{O}_Y$-module, we can define
   $$\phi^*\mathcal{F} = \phi^{-1}\mathcal{F} \otimes_{\phi^{-1}(\mathcal{O}_Y)} \mathcal{O}_X.$$
   (This definition should be interpreted with *sheafification*, as usual for operations on sheaves.) This amounts to forcing $\phi^*\mathcal{F}$ to be a sheaf of $\mathcal{O}_X$-modules "by hand." Show that if $\mathcal{F}$ is the sheaf of sections of F then $\phi^*\mathcal{F}$ really is the sheaf of sections of ϕ^*F. (It is probably a good idea to first consider the simple case of a covering map, say 2-1.)

Exercise 4

(For those who like counterexamples.) One might have tried to define the sheafification $\mathcal{F}^+$ of a presheaf $\mathcal{F}$ by taking $\mathcal{F}^+(U)$ to be the space of "discontinuous sections" $s \in \prod_{x \in U} \mathcal{F}_x$, subject to the condition that there exist a covering of U by open sets U_i with $f_i \in \mathcal{F}(U_i)$, $f_i|_{U_{ij}} = f_j|_{U_{ij}}$, and $(f_i)_x = s_x$. This doesn't quite work if your presheaf is crazy enough: since $\mathcal{F}$ is only a presheaf, $\mathcal{F}^+$ may involve coverings by sections that agree on stalks but don't agree on intersections! Read and understand the counterexample described at

`http://mathoverflow.net/questions/31372/`

Naturally, this kind of thing won't bother us in the rest of the course.