Complex Geometry: Exercise Set 2

Exercise 1

Show that the space of holomorphic 1-forms on the torus X_τ is 1-dimensional. (We described one such 1-form in class, dz, so the issue is to be sure you understand what that 1-form is, and to show that there are no others.)

Exercise 2

1. Show that the space of holomorphic sections of the line bundle $\mathcal{O}(n)$ over $\mathbb{CP}^1$ has dimension $n+1$ for $n \geq 0$, and dimension 0 for $n < 0$. (One direct way to do it is to use the description in terms of two patches.)
2. Show that the holomorphic tangent bundle of $\mathbb{CP}^1$ is isomorphic (as a holomorphic line bundle) to $\mathcal{O}(2)$.
3. (For those who like Lie algebras.) From the previous two parts, it follows that the space of holomorphic vector fields on $\mathbb{CP}^1$ is 3-dimensional. Taking brackets we thus obtain a complex 3-dimensional Lie algebra. Write out the Lie algebra structure explicitly. Show that this Lie algebra is isomorphic to $sl(2,\mathbb{C})$.

Exercise 3

Suppose (X, I) is an almost complex manifold. The *Nijenhuis tensor* N is a tensor field of type $(2,1)$, i.e. a section of $(T^*X)^{\otimes 2} \otimes TX = \mathrm{Hom}(TX^{\otimes 2}, TX)$, given by

$$N(v,w) = [v,w] + I[Iv,w] + I[v,Iw] - [Iv,Iw]$$

for two vector fields v, w on X.

1. Show that the above formula indeed defines a tensor, i.e. $N(v,w)$ at a point $x \in X$ only depends on the values of v and w at that point, not on their extension to vector fields on X; this amounts to checking that $N(fv,w) = fN(v,w)$ and $N(v,fw) = fN(v,w)$ for any function f on X.
2. Show that I is integrable if and only if $N = 0$.

Exercise 4

1. Suppose $\mathcal{L}_1$ and $\mathcal{L}_2$ are two holomorphic line bundles on a complex manifold X, of dimension at least 2. Suppose that for some point $x \in X$, $\mathcal{L}_1|_{X\setminus\{x\}} \simeq \mathcal{L}_2|_{X\setminus\{x\}}$ are isomorphic. Show that $\mathcal{L}_1 \simeq \mathcal{L}_2$. (You will probably need Hartogs' Theorem, Proposition 1.1.4 of Huybrechts.)
2. Show by example that the same is not true if X is of dimension 1. (For example, $\mathcal{O}(0)$ and $\mathcal{O}(1)$ over $\mathbb{CP}^1$ are not isomorphic — why? — but their restrictions to $\mathbb{CP}^1 \setminus \{x\}$ are isomorphic.)