Complex Geometry: Exercise Set 1

These exercises are mostly meant to solidify your understanding of the basic definitions.

Exercise 1

Let $P : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with only simple zeroes. Consider the space $X = \{P(x_1) - x_2^2 = 0\} \subset \mathbb{C}^2$. We said in class that this is a 1-dimensional complex manifold, with a bit of hand-waving about the implicit function theorem. In this exercise we prove this more directly.

1. Let $\mathbf{x}_0$ be a point of X with $x_2 \neq 0$. Show that x_1 is a good local coordinate in a neighborhood of $\mathbf{x}_0$, i.e. x_1 gives a 1-1 map between a neighborhood of $\mathbf{x}_0$ in X and an open set in $\mathbb{C}$.
2. Let $\mathbf{x}_0$ be a point of X with $x_2 = 0$. Show that x_2 is a good local coordinate in a neighborhood of this point.
3. Describe a holomorphic atlas on X.

Exercise 2

For τ in the upper half-plane we defined a structure of complex manifold on

$$X_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$$

induced from that on $\mathbb{C}$.

1. Show that $X_\tau \simeq X_{\tau+1}$ and $X_\tau \simeq X_{-1/\tau}$.

   (It follows that $X_\tau \simeq X_{\tau'}$ whenever $\tau' = \frac{a\tau+b}{c\tau+d}$, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$.)
2. For general τ, τ' in the upper half-plane, construct an explicit diffeomorphism $X_\tau \to X_{\tau'}$. What does it look like relative to complex coordinates for X_τ and $X_{\tau'}$? (The point of this is just to see explicitly that this diffeomorphism is not holomorphic.)

Exercise 3

1. Show that $\mathbb{CP}^1$ is homeomorphic to S^2 (indeed diffeomorphic, with the usual smooth structure on S^2.) In particular $\mathbb{CP}^1$ is simply connected.
2. Show that there are no non-constant holomorphic maps from $\mathbb{CP}^1$ to X_τ.

Exercise 4

This exercise asks you to fill in the details of some constructions of new vector bundles from old. If you are not comfortable with the language of functor and category, ignore questions 1 and 5; in that case you will have to do the other questions in a more ad hoc fashion.

1. Let Vect be the category of complex vector spaces, and Vect_X the category of C^∞ complex vector bundles over X. For any $x \in X$ there is a functor $R_x : \mathrm{Vect}_X \to \mathrm{Vect}$

which takes the fiber over x. Suppose given a *smooth* functor $S : \mathrm{Vect} \to \mathrm{Vect}$, i.e. a functor such that the maps $S : \mathrm{Hom}(A, B) \to \mathrm{Hom}(S(A), S(B))$ are C^∞. Show that S induces a functor $S_X : \mathrm{Vect}_X \to \mathrm{Vect}_X$, such that $R_x \circ S_X = S \circ R_x$. (One way to describe S_X is that it "acts by S on each fiber.") Do similarly if S is a functor from Vect to Vect^{op} (contravariant functor).

2. For example, taking S to be the complex-conjugation functor, S_X is the functor which takes a vector bundle E over X to its conjugate bundle $\bar{E}$. Describe this concretely: given a description of E by transition functions, give a description of $\bar{E}$ by transition functions.
3. For example, taking S to be the (contravariant) dualization functor, S_X is the functor which takes a vector bundle E over X to its dual bundle E^*. Describe this concretely: given a description of E by transition functions, give a description of E^* by transition functions.
4. A slight extension of this discussion to functors $\mathrm{Vect}^n \to \mathrm{Vect}$ allows us to define the direct sum $E \oplus F$ and tensor product $E \otimes F$ of two vector bundles. Define them. How are the transition functions of $E \oplus F$ and $E \otimes F$ related to those of E and F?
5. In all of the preceding we used C^∞ vector bundles. We can similarly consider the category $\mathrm{Vect}_X^{\mathrm{hol}}$ of holomorphic vector bundles over X. Suppose given a *holomorphic* functor $S : \mathrm{Vect} \to \mathrm{Vect}$, i.e. one for which the induced maps $S : \mathrm{Hom}(A, B) \to \mathrm{Hom}(S(A), S(B))$ are holomorphic. Show that in this case S induces a functor $S_X^{\mathrm{hol}} : \mathrm{Vect}_X^{\mathrm{hol}} \to \mathrm{Vect}_X^{\mathrm{hol}}$, such that $R_x \circ S_X^{\mathrm{hol}} = S \circ R_x$.
6. Concretely, if E and F are holomorphic, which of E^*, $\bar{E}$ and $E \oplus F$ carry natural holomorphic structures?

Exercise 5

1. Suppose $\mathcal{L}$ is a holomorphic line bundle with a holomorphic section which is nowhere vanishing. Show that $\mathcal{L}$ is the trivial holomorphic line bundle.
2. If $\mathcal{L}$ is a holomorphic line bundle, show that $\mathcal{L} \otimes \mathcal{L}^*$ is the trivial holomorphic line bundle.
3. If E is a holomorphic vector bundle, show that $E \otimes E^*$ has a canonical holomorphic section. (One could also call this vector bundle $\mathrm{Hom}(E, E)$.)
4. Suppose X is a compact connected complex manifold. If $\mathcal{L}$ is a holomorphic line bundle with a nonzero holomorphic section and $\mathcal{L}^*$ also has a nonzero holomorphic section, show that $\mathcal{L}$ is the trivial holomorphic line bundle.