Mathieu equation

$$\left(-\frac{\hbar^2}{2}\partial_x^2 + \cos x - E\right)\psi(x) = 0$$

[Schrödinger equation with periodic potential]

Parallel transport of this equation defines a flat $SL(2)$-connection ∇ over S^1

Q: what is its <u>monodromy</u> M?

more concrete Q: what are the <u>eigenvalues</u> of monodromy? ("quasimomenta": $\psi(x+2\pi) = \lambda\psi(x)$)

When $E \gg 1$, <u>approx</u> neglect $\cos x \rightsquigarrow$ $\psi_\pm(x) = e^{\pm i\frac{\sqrt{2E}}{\hbar}x}$ give a basis of solutions

$\rightsquigarrow$ eigenvalues $\lambda_\pm = e^{\pm 2\pi i\frac{\sqrt{2E}}{\hbar}}$, trace $= 2\cos\left(\frac{2\pi\sqrt{2E}}{\hbar}\right)$

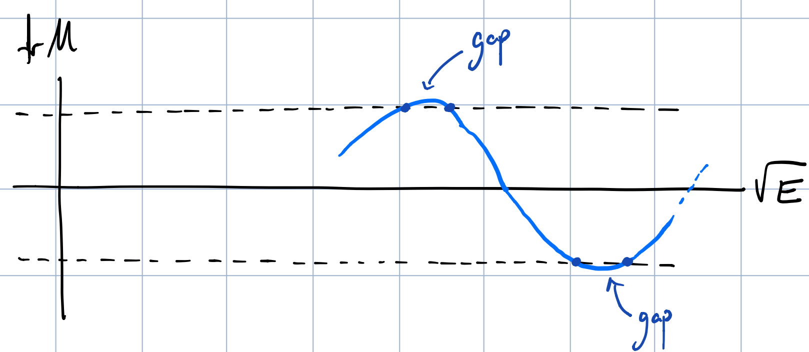
$|\lambda_\pm| = 1$ for all E

The <u>exact</u> behavior is more interesting:

"band gaps" where $|\mathrm{tr}\, M| > 2$, ie $|\lambda| \neq 1$,

of size $\Delta E \simeq \exp\left(-\frac{2\sqrt{2E}\log E}{\hbar}\right)$

"nonperturbative"



How to study this in a systematic way?

complexify everything, pass to coordinate $z = e^{ix}$

then have an eq. of the form

$$\left(\hbar^2\partial_z^2 + P(z)\right)\psi(z) = 0 \qquad P(z) = \frac{1}{z^3} - \frac{2E - \cancel{\frac{1}{4}\hbar^2}}{z^2} + \frac{1}{z}$$

"exact WKB" method: construct solutions in the form

$$\psi = \exp\left(\hbar^{-1}\int\lambda\, dz\right)$$

Then $\partial_z\psi - \hbar^{-1}\lambda\psi = 0 \Rightarrow \psi$ is section of a flat $\mathbb{C}^\times$-bundle

λ obeys <u>Riccati equation</u> $\lambda^2 + P + \hbar\partial_z\lambda = 0.$

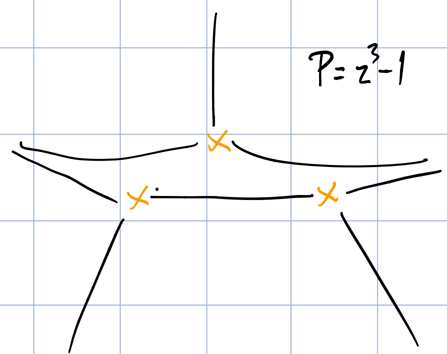
Can construct such λ in formal series, without integration:

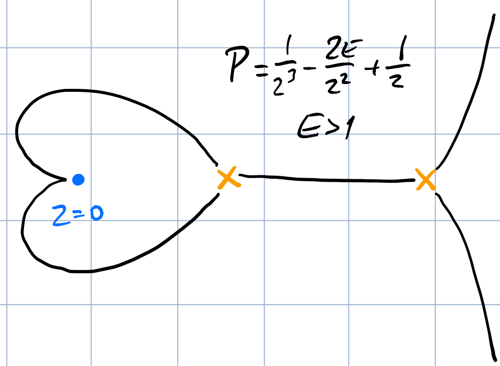
$$\lambda = \sqrt{-P} - \hbar \frac{P'}{4P} + \hbar^2 \sqrt{-P}\, \frac{5P'^2 - 4PP''}{32P^3} + \cdots$$

naturally lives on the spectral curve $\Sigma = \{ y^2 + P(z) = 0 \}$ double cover

$$\Sigma \downarrow \mathbb{C}$$

This series is only asymptotic but it admits Borel summation for $\hbar > 0$, away from "Stokes graph"/"spectral network" $\mathcal{W}(P)$ (critical graph of $P(z)\,dz^2$)





$\Rightarrow$ ∇ reduces to a flat connection ∇^{ab} in $\mathcal{L} \to \Sigma$ away from $\mathcal{W}(P)$

(picks out preferred lines in solution space on each domain)

and, $\exists$ canonical gluing that extends ∇^{ab} over the full Σ,

$\rightsquigarrow$ flat connection ∇^{ab} in $\mathcal{L} \to \Sigma$

Let $X_\gamma = (\mathrm{hol}_\gamma \nabla^{ab}) \in \mathbb{C}^\times$

These quantities turn out to have cool properties:

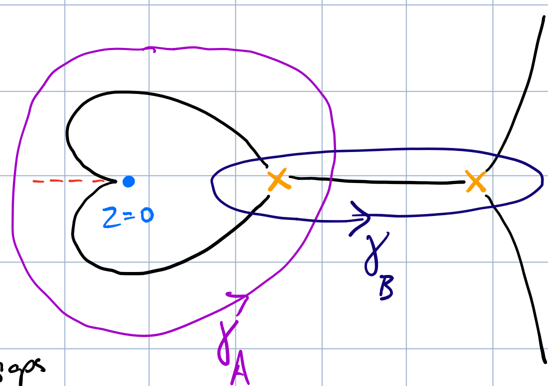
1) $\hbar \to 0$ asymptotic series expansion computable from λ, Borel summable (more or less)

eg $X_\gamma \sim \exp(\hbar^{-1} Z_\gamma)$ $\quad Z_\gamma = \oint_\gamma \sqrt{-P}\, dz$

2) they are invariants of ∇, eg here

$$X_A = \sqrt{\frac{\psi \wedge M\psi'}{M\psi \wedge \psi'}} \quad \left(\psi, \psi' \text{ exp. decay at } 0, \infty\right)$$

$$X_B = \cdots$$



3) $\mathrm{Tr}(M) = (X_A + X_A^{-1})\sqrt{1 + X_B}$

$|X_A| = 1$, so X_B responsible for band gaps

$\Rightarrow$ compute their size $\sim \exp\left(\frac{1}{2\hbar} Z_B\right)$

4) they can be computed by a TBA-like integral equation

T_3 equation

Now consider instead equations of the form:

$$\boxed{\left[\partial_z^3 + \hbar^{-2} P_2(z)\,\partial_z + \left(\hbar^{-3} P_3(z) + \tfrac{1}{2}\hbar^{-2} P_2'(z)\right)\right] \psi(z) = 0}$$

specific case: $P_2 = \dfrac{9\hbar^2}{(z^3-1)^2}$ $\quad P_3 = -\dfrac{u}{(z^3-1)^2}$

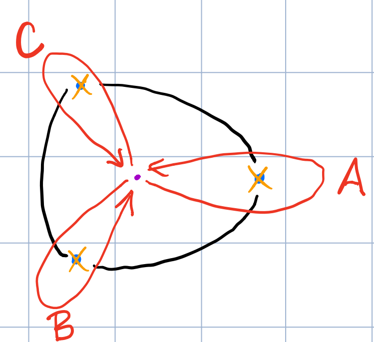
$\rightsquigarrow$ flat $SL(3)$-connection ∇ over $\mathbb{CP}^1$ w/ reg sing at $1, \omega, \omega^2$

Say we want to study ∇ when $u > 0$.

What are the "good coordinates"?

As before, look for solutions in the form $\psi = \exp \hbar^{-1} \int \lambda$ — now live on spectral cover

$$\Sigma = \{\lambda^3 + P_2 \lambda + P_3 = 0\}$$
$$\downarrow 3:1$$
$$\mathbb{CP}^1$$



What are they?

In interior domain:

basis (ψ_1, ψ_2, ψ_3)

obeying the conditions

$C\psi_1, B^{-1}\psi_2 \in \langle \psi_1, \psi_2 \rangle$

$A\psi_2, C^{-1}\psi_3 \in \langle \psi_2, \psi_3 \rangle$

$B\psi_3, A^{-1}\psi_1 \in \langle \psi_3, \psi_1 \rangle$

So eg $A = \begin{pmatrix} * & 0 & * \\ * & * & * \\ * & * & * \end{pmatrix}$, $A^{-1} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ * & * & * \end{pmatrix}$

Similarly $B^{\pm 1}, C^{\pm 1}$

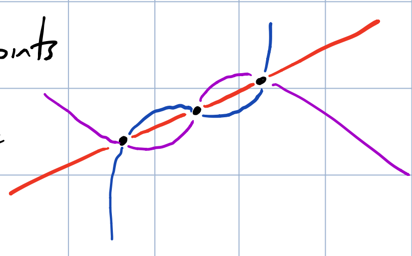
How many such bases exist? "$4 + \infty$"

— look at pencil of cubic curves $E_t = \{\psi \wedge C\psi \wedge A^{-1}\psi + t \cdot \psi \wedge C^{-1}\psi \wedge A\psi = 0\} \subset \mathbb{CP}^2$

This has 3 base-points

one E_t is triple line

4 other E_t are nodal



and the desired projective bases are in 1-1 correspondence with the singular points $(\psi_1 = \psi)$

Then $\mathcal{X}_A = \dfrac{\psi_2 \wedge \psi_3 \wedge \psi_1}{C^{-1}\psi_3 \wedge A\psi_2 \wedge \psi_1}$, $\mathcal{X}_B =$ (more complicated)

So these are the "good coordinates" for analyzing ∇ along the ray $u > 0$.

eg computing traces of monodromies...

Can look at "spectral problems" like $\mathcal{X}_A = \mathcal{X}_B = 0$, ...