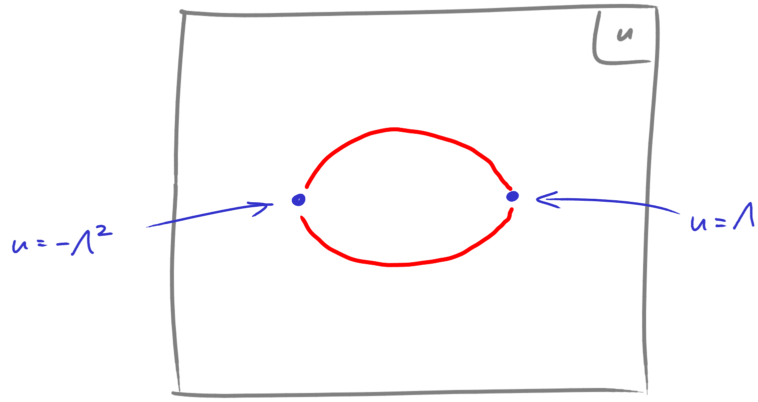
Wall-Crossing (2)

Ex $\mathcal{N}=2$ SYM, $G=SU(2)$.

$\mathcal{B}=\mathbb{C}$ $\quad u=\langle \mathrm{Tr}\,\varphi^2\rangle$



$$\gamma = q\gamma_e + p\gamma_m$$

$$Z_{\gamma_m} = \frac{i}{4}\Lambda(\alpha-1)\,{}_2F_1\left(\tfrac{3}{4},\tfrac{3}{4},2;1-\alpha\right)$$
$$Z_{\gamma_e} = \sqrt{2}\,\Lambda\,\alpha^{1/4}\,{}_2F_1\left(-\tfrac{1}{4},\tfrac{1}{4},1;\alpha\right) \qquad \alpha = u^2/\Lambda^4$$

NB: $Z_{\gamma_e}, Z_{\gamma_m}$ are <u>not single-valued</u>! e.g. under monodromy around $u=\Lambda^2$

have
$$Z_{\gamma_m} \to Z_{\gamma_m}$$
$$Z_{\gamma_e} \to Z_{\gamma_e} - 2Z_{\gamma_m} = Z_{\gamma_e - 2\gamma_m} \qquad \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

This reflects the fact that we must use $SL(2,\mathbb{Z})$ E/M duality transformations to glue together the <u>patches</u> to get a <u>global</u> picture of the IR physics.

Wall-crossing formula

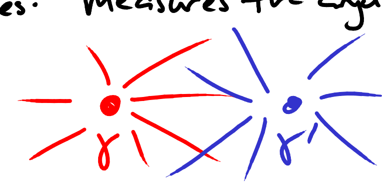
Written down first by Kontsevich–Soibelman in context of "generalized Donaldson–Thomas invariants."

A slight rephrasing:

Consider an algebra w/ generators $X_\gamma \qquad \gamma\in\Gamma$

$$X_\gamma X_{\gamma'} = (-1)^{\langle\gamma,\gamma'\rangle} X_{\gamma+\gamma'} \qquad \text{("twisted torus algebra")}$$

$\underset{\mathbb{Z}}{\underset{\uparrow}{\langle\gamma,\gamma'\rangle}}$ is "DSZ inner product" on e/m charges: measures the angular momentum in the crossed em fields

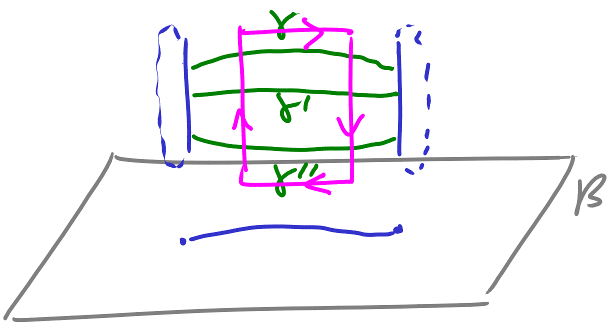


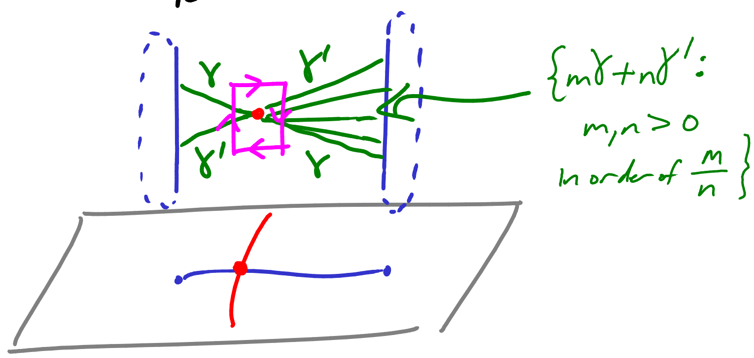
Define for each γ an <u>automorphism</u> of this algebra:

$$K_\gamma : \; X_{\gamma'} \longrightarrow X_{\gamma'} (1 - X_\gamma)^{\langle \gamma', \gamma \rangle}$$

Now, consider $\mathcal{B} \times S^1$. On this space, mark a codim-1 locus C_γ each charge γ w/ $\Omega(\gamma) \neq 0$,

$$C_\gamma = \{ (\underset{\mathcal{B}}{\underset{\cap}{u}} , \arg -Z_\gamma(u)) \}$$

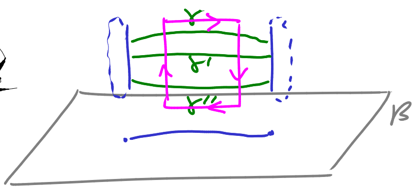




Now consider <u>any closed loop</u> $P \subset \mathcal{B} \times S^1$.

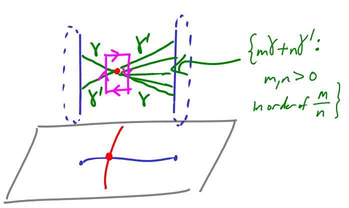
Define $\displaystyle A(P) = \prod_{P \cap (\cup C_\gamma)} K_\gamma^{\pm \Omega(\gamma)}$ ($\pm$ depends which way we cross)

Ex



$$K_{\gamma''} K_{\gamma'} K_\gamma K_\gamma^{-1} K_{\gamma'}^{-1} K_{\gamma''}^{-1} = \mathbb{1}.$$

Ex



$$K_{\gamma'} K_\gamma \prod_{m/n \nearrow} K_{m\gamma + n\gamma'}^{-\Omega(m\gamma + n\gamma')} = \mathbb{1}$$

K-S statement

$$K_{\gamma'} K_\gamma = \prod_{m/n \searrow} K_{m\gamma + n\gamma'}^{\Omega(m\gamma + n\gamma')}$$

Claim: this <u>determines</u> the $\Omega(m\gamma + n\gamma')$ on the RHS!

Ex If $\langle \gamma, \gamma' \rangle = 1$ then $\underbrace{K_{\gamma'} K_\gamma}_{\substack{\text{2 BPS states} \\ \text{(hypermultiplet)}}} = \underbrace{K_\gamma K_{\gamma+\gamma'} K_{\gamma'}}_{\substack{\text{3 BPS states} \\ \text{(all hypermultiplets)}}}$

e.g. to check this, act with both sides on X_γ:

LHS: $X_\gamma \xrightarrow{K_\gamma} X_\gamma \xrightarrow{K_{\gamma'}} (1 - X_{\gamma'}) X_\gamma = X_\gamma + X_{\gamma+\gamma'}$

RHS: $X_\gamma \xrightarrow{K_{\gamma'}} X_\gamma + X_{\gamma+\gamma'} \xrightarrow{K_{\gamma+\gamma'}} X_\gamma + X_{2\gamma+\gamma'} + X_{\gamma+\gamma'}$

$$\xrightarrow{K_\gamma} X_\gamma + (1 - X_\gamma)^{-1} (X_{2\gamma+\gamma'} + X_{\gamma+\gamma'})$$

$$= X_\gamma + X_{\gamma+\gamma'}$$

Ex If $\langle \gamma, \gamma' \rangle = 2$ then

$$\underset{\text{monopole}}{K_{\gamma'}}\, \underset{\text{dyon}}{K_\gamma} = \underbrace{\left(\prod_{n=1}^{\infty} K_{n\gamma + (n-1)\gamma'} \right)}_{\text{dyons}} \underset{\text{W boson}}{K_{\gamma+\gamma'}^{-2}} \underbrace{\left(\prod_{n=\infty}^{1} K_{(n-1)\gamma + n\gamma'} \right)}_{\text{dyons}}$$

This is exactly what we need for $\mathcal{N}=2$ SYM with $G = SU(2)$!

Why is the formula true? [Gaiotto-Moore-Neitzke]

Need to understand the physics of X_γ and K_γ.

Consider a SUSic line operator ($\frac{1}{2}$-BPS) in our $\mathcal{N}=2$ theory.

[**Ex** in abelian gauge theory, the usual Wilson line op. is

$$L = \exp\left[i \int_P A \right].$$

SUSY version of this:

$$L_\vartheta = \exp\left[i \int_p A + e^{-i\vartheta}\varphi \, ds + e^{i\vartheta}\bar{\varphi}\, ds \right]$$

If p is a <u>straight</u> <u>timelike path</u> in $\mathbb{R}^{3,1}$ then L_ϑ preserves $\frac{1}{2}$ of the SUSY: the generators we called $R, \bar{R}$ — just the same ones preserved by a BPS particle w/ $\vartheta = -\arg Z$.

So inserting L_ϑ is like inserting a "very heavy BPS particle with phase ϑ"

Define a general SUSic line op L_ϑ to be one that preserves $R, \bar{R}$.

e.g. nonabelian Wilson line, 't Hooft line...

In the presence of L_ϑ the Hilbert space is modified: $\mathcal{H} \rightsquigarrow \mathcal{H}_{L_\vartheta}$

BPS bound for states of charge γ becomes

$$M \geq \mathrm{Re}(e^{i\vartheta} Z_\gamma)$$

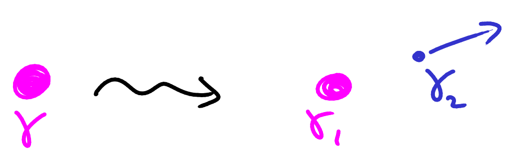
Now consider framed BPS states saturating this bound (annihilated by $R, \bar{R}$). Counted by

$$\overline{\underline{\Omega}}(L_\vartheta, \gamma) = \mathrm{Tr}_{\mathcal{H}'_{L_\vartheta}} (-1)^{2J_3}$$

Like $\Omega(\gamma)$, can ask how $\overline{\underline{\Omega}}(L_\vartheta, \gamma)$ behaves under deformations.

Invariant, unless we have <u>mixing</u> with the <u>continuum</u>:

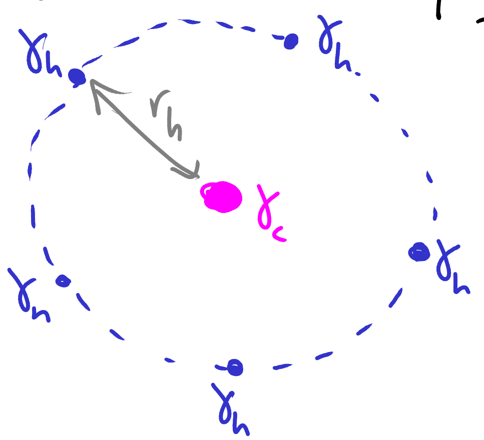
framed BPS state can decay by emitting an ordinary ("vanilla") BPS state.



$\gamma = \gamma_1 + \gamma_2$

This can happen when the 2 constituents have aligned central charges:
i.e. when $\vartheta = -\arg Z_{\gamma_2}$

Moreover, in this case we (mostly Denef) can really calculate how $\overline{\Omega}(L_\vartheta, \gamma)$ jumps: the states which appear/disappear at $\vartheta = -\arg Z_\gamma$ have a nice classical picture, e.g.



$\gamma_c + 5\gamma_h$

$$r_h = \frac{\langle \gamma_h, \gamma_c \rangle}{2\,\mathrm{Im}(e^{i\vartheta} Z_{\gamma_h})}$$

As $\vartheta \to -\arg Z_{\gamma_h}$ these states disappear/appear in $\mathcal{H}^1_{L_\vartheta,\gamma}$

We can calculate exactly how many such states:

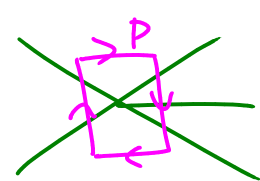
Define a generating f^n
$$F_{L_\vartheta} = \sum_\gamma \overline{\Omega}(L_\vartheta, \gamma)\, X_\gamma$$

As ϑ crosses $-\arg Z_{\gamma_h}$, F_{L_ϑ} is transformed by

$$X_\gamma \longrightarrow X_\gamma (1 - X_{\gamma_h})^{\pm \Omega(\gamma_h)\langle \gamma, \gamma_h \rangle} \quad \text{i.e. } K_{\gamma_h}^{\pm \Omega(\gamma_h)} \,!$$

But, when we travel around a closed loop in param. space,
F_{L_ϑ} must come back to itself...

i.e.



i.e. $\prod \mathcal{K}_\gamma^{\pm\Omega(\gamma)} = A(P)$ preserves F_{L_ϑ}

If the theory has "enough" line ops in it, this implies $A(P) = \mathbb{1}$.

That is the WCF that we wanted to prove.