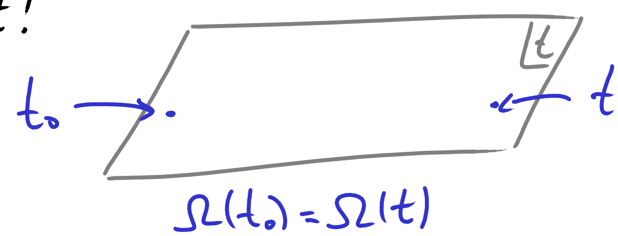
Wall-Crossing (1)

What is Wall-Crossing?

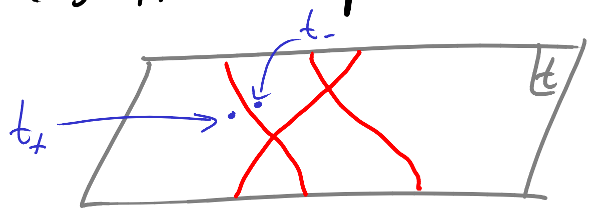
Study a quantum field theory depending on params t
(e.g. couplings or choice of vacuum)

A popular game, esp. in SUSY QFT: Find quantities $\Omega(t)$ which are indep of t.
Then, can calculate at some $t = t_0$ (e.g. at weak coupling) and extrapolate to all other t!



Useful e.g. for testing dualities
(also for studying black holes...)

Often we meet a (slightly) more complic. situation: $\Omega(t)$ is only piecewise indep. of t
it jumps at some codim-1 loci (walls)



Then, we may ask: how are $\Omega(t_+)$ and $\Omega(t_-)$ related? Wall-crossing.

Recently this pb has been solved in imp^t example: $\mathcal{N}=2$ SUSY FT in $d=4$,
where $\Omega(\gamma, t)$ is counting "# of BPS states with charge γ."

My aim:
- to explain the solution
- to explain why it is true
- to explain some cool things you can use it for (construction of HK metrics)

$\mathcal{N}=2$ SUSY in $d=4$

Unlike non-SUSY or $\mathcal{N}=1$ SUSY, $\mathcal{N}=2$ SUSY f.t. typically have a cts moduli space of vacua. It has several branches: we'll focus on the "Coulomb branch" $\mathcal{B}$.

Ex $\mathcal{N}=2$ SYM with gauge group $G = SU(2)$.

Fields: gauge field A^μ, complex scalar φ, fermions ψ } in adjoint of $su(2)$

Action includes a potential for φ, $V = \mathrm{Tr}\,[\varphi, \varphi^\dagger]^2$

flat directions: take $\varphi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ $a \in \mathbb{C}$

$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \sim \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}$: vacua parameterized by $u = \frac{1}{2}\mathrm{Tr}\,\varphi^2 = a^2$

So Coul. branch $\mathcal{B}$ (classically) is just the complex u-plane

At any $u \neq 0$, the IR physics is U(1) gauge theory ($\mathcal{N}=2$).
(b/c φ Higgses $SU(2) \to U(1)$).

- This is what happens generally: $\mathcal{B}$ is a complex manifold and the IR physics at a generic point in $\mathcal{B}$ is abelian $\mathcal{N}=2$ gauge theory, gauge gp $U(1)^r$.

- States carry electromagnetic charges under $U(1)^r$:

  charge lattice $\Gamma \simeq \mathbb{Z}^{2r}$

  (plus possible flavor charges, neglect these)

  $$\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma \qquad \mathcal{H}^1 = \bigoplus_{\gamma \in \Gamma} \mathcal{H}^1_\gamma$$

  1-particle

- $\mathcal{H}$ is rep. of SUSY algebra $\mathcal{A}$: $iso(3,1) \oplus (\text{odd part}) \oplus \mathbb{C}$

  Z

  Odd part generated by spinors Q^A $A = 1,2$

  $$\{Q^A, \overline{Q}^B\} = \delta^{AB} P$$
  $$\{Q^A, Q^B\} = \varepsilon^{AB} Z$$

Z is the "central charge": it acts by a scalar Z_γ on each $\mathcal{H}_\gamma$ with

$$Z_{\gamma+\gamma'} = Z_\gamma + Z_{\gamma'}$$

$Z : \Gamma \to \mathbb{C}$ homomorphism

- Look at representations of $\mathcal{A}$ on massive 1-particle states.

First, w/o SUSY: Wigner says massive reps of $iso(3,1)$ classified by mass M, spin j

(decompose according to mass, then choose rest frame and decompose under "little group" $so(3)$ preserving rest frame)

Super version of Wigner's classification:

still have Casimir M

and new central charge Z

choose rest frame, decompose under "little supergroup": $so(3)$ and Q's.

A good trick for studying this supergroup:

$$R = \frac{1}{\sqrt{2}}\left[e^{-i\vartheta/2} Q^1 + e^{i\vartheta/2} \overline{Q}^2\right]$$
$$T = \frac{1}{\sqrt{2}}\left[e^{-i\vartheta/2} Q^1 - e^{i\vartheta/2} \overline{Q}^2\right]$$

$\Rightarrow$

$$\{R, \overline{R}\} = M - \mathrm{Re}(e^{i\vartheta} Z)$$
$$\{T, \overline{T}\} = M + \mathrm{Re}(e^{i\vartheta} Z)$$
$$\{R, \overline{T}\} = i\,\mathrm{Im}(e^{i\vartheta} Z)$$
$$\{\overline{R}, T\} = -i\,\mathrm{Im}(e^{i\vartheta} Z)$$

Choose $\vartheta = -\arg(Z)$, then $\mathrm{Re}(e^{i\vartheta} Z) = |Z|$, $\mathrm{Im}(e^{i\vartheta} Z) = 0$

So
$$\{R, \overline{R}\} = M - |Z|$$
$$\{T, \overline{T}\} = M + |Z|$$
all others zero.

In particular: $\|R\psi\|^2 + \|\overline{R}\psi\|^2 = \langle\psi|\{R,\overline{R}\}|\psi\rangle = \langle\psi|(M-|Z|)|\psi\rangle = (M-|Z|)\|\psi\|^2$

$$\Rightarrow M - |Z| \geq 0$$

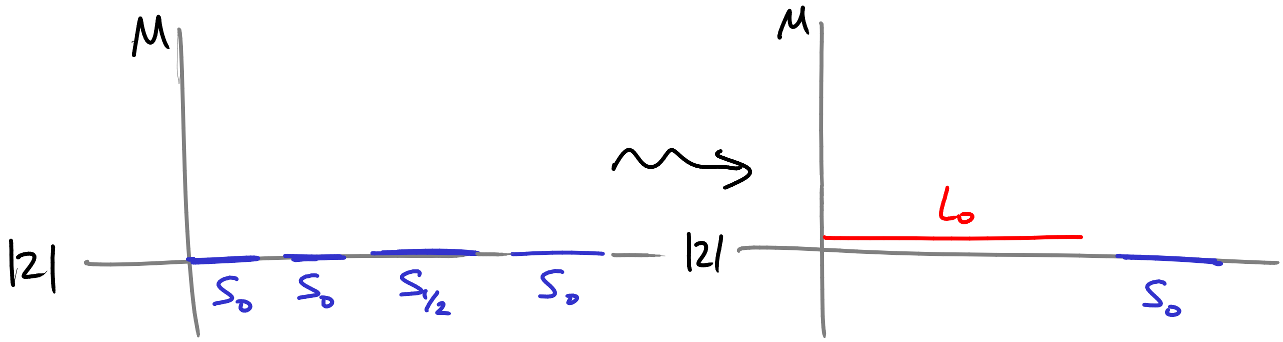
$$M \geq |Z| \qquad \text{"BPS bound"}$$

If $M = |Z|$ then $\{R, \overline{R}\} = 0$ "BPS state"

So massive reps of $\mathcal{A}$ come in two types:

- $M > |Z|$: "long representations" — at rest, $[j] \otimes \overbrace{(2[0] \oplus [\tfrac{1}{2}])}^{\text{from } T} \otimes \overbrace{(2[0] \oplus [\tfrac{1}{2}])}^{\text{from } R} = L_j$
- $M = |Z|$: "short/BPS reps" — " " $[j] \otimes (2[0] \oplus [\tfrac{1}{2}]) = S_j$

Because the S_j are short, they are <u>protected</u>: e.g. S_0 cannot deform into any L_j.
So might think the # of copies of S_j in $\mathcal{H}^1_\gamma$ is a deformation invariant.
Not quite: e.g. $2 S_0 \oplus S_{1/2} \simeq L_0$ so this combination <u>could</u> disappear.



Instead, we define

$$\Omega(\gamma) = -\frac{1}{2} \mathrm{Tr}_{\mathcal{H}^1_{\gamma,\mathrm{rest}}} (J_3)^2 (-1)^{2J_3}$$

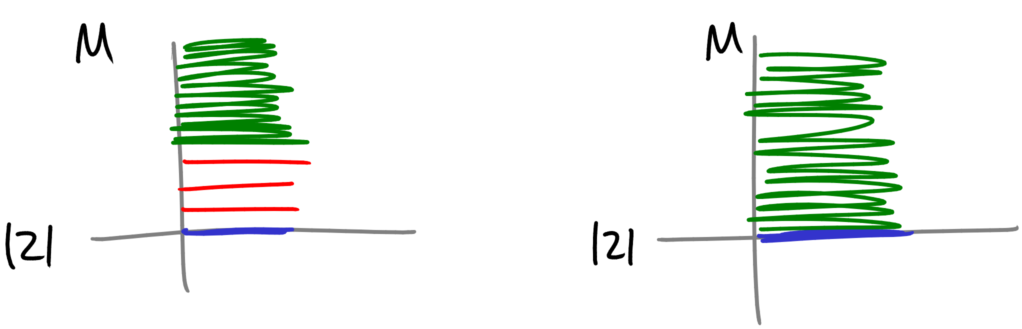
This counts the rep. S_j with weight $(-1)^{2j}(2j+1)$ e.g. $+1$ from S_0

-2 from $S_{1/2}$

$\vdots$

$\Omega(\gamma)$ is <u>invariant</u> under arbitrary deformations of $\mathcal{H}^1_{\gamma,\mathrm{rest}}$.

- <u>But</u>, $\Omega(\gamma)$ is <u>not</u> inv't under arb. def. of the field theory!
  It becomes <u>ill defined</u> when $\mathcal{H}^1_\gamma$ mixes with the <u>multiparticle spectrum</u>.



OK NOT OK

When does this mixing happen? Look at a 2-particle state: $\gamma_1 + \gamma_2 = \gamma$

Say it has $M = |Z_\gamma|$ (mixing).

$$|Z_{\gamma_1} + Z_{\gamma_2}| = |Z_{\gamma_1+\gamma_2}| = |Z_\gamma| = M \geq M_1 + M_2 \geq |Z_{\gamma_1}| + |Z_{\gamma_2}|$$

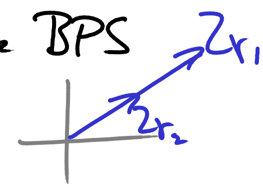
so $\quad |Z_{\gamma_1} + Z_{\gamma_2}| \geq |Z_{\gamma_1}| + |Z_{\gamma_2}|$

But the Δ inequality says

$$|Z_{\gamma_1} + Z_{\gamma_2}| \leq |Z_{\gamma_1}| + |Z_{\gamma_2}| .$$

It's consistent only if all these inequal. are actually equalities:

- $M_1 = |Z_{\gamma_1}|$, $M_2 = |Z_{\gamma_2}|$ — the constituents must be BPS
- $|Z_{\gamma_1} + Z_{\gamma_2}| = |Z_{\gamma_1}| + |Z_{\gamma_2}|$ — $Z_{\gamma_1}/Z_{\gamma_2} \in \mathbb{R}_+$



So: $\Omega(\gamma)$ is well def. and deformation inv^t,
except at the "walls of marginal stability"
where $\dfrac{Z_{\gamma_1}}{Z_{\gamma_2}} \in \mathbb{R}_+$, $\gamma_1 + \gamma_2 = \gamma$, $\Omega(\gamma_1) \neq 0$, $\Omega(\gamma_2) \neq 0$

Q: How does the collection $\{\Omega(\gamma)\}_{\gamma \in \Gamma}$ jump when we cross a wall?

Ex $\mathcal{N}=2$ SYM, $G = SU(2)$.

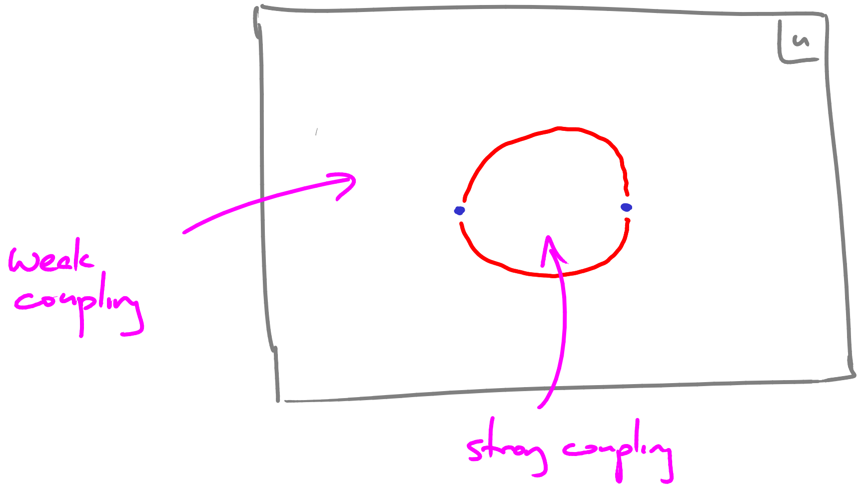
$\mathcal{B}$ = complex plane, coordinate $u = \frac{1}{2}\langle \mathrm{Tr}\, \varphi^2 \rangle$. $\quad \Lambda$ = QCD scale.

Try writing charges as $\gamma = q\gamma_e + p\gamma_m$

"electric" $\qquad$ "magnetic"

Then $Z_{\gamma_m} = \frac{i}{4}\Lambda(\alpha - 1)\, {}_2F_1\left(\frac{3}{4}, \frac{3}{4}, 2; 1-\alpha\right) \qquad \left(\alpha = \frac{u^2}{\Lambda^4}\right) \qquad$ [SW]

$\qquad Z_{\gamma_e} = \sqrt{2}\,\Lambda\,\alpha^{1/4}\, {}_2F_1\left(-\frac{1}{4}, \frac{1}{4}, 1; \alpha\right)$



singularities at $u = \pm\Lambda^2$.

wall where $\frac{Z_{\gamma_e}}{Z_{\gamma_m}} \in \mathbb{R}$

At strong coupling: $\Omega(\pm\gamma_m) = 1$ (monopole)

$\Omega(\pm(2\gamma_e - \gamma_m)) = 1$ (dyon)

other $\Omega(\gamma) = 0$

At weak coupling: $\Omega(\pm 2\gamma_e) = -2$ (W boson)

$\Omega(n\gamma_e \pm \gamma_m) = 1 \quad \forall n \in \mathbb{Z}$ (dyons/monopoles)

other $\Omega(\gamma) = 0$