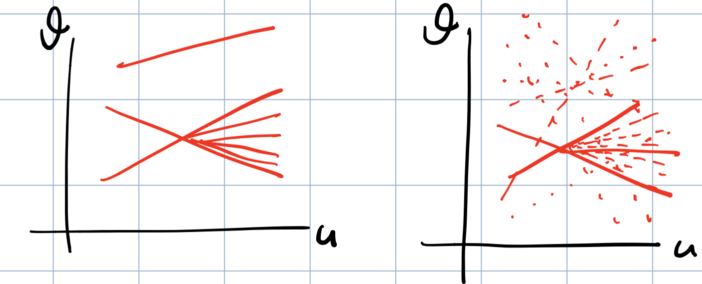
Last time:

$$\left[\begin{array}{l} \cdot\ C \text{ compact Riemann surface} \\ \cdot\ N \geq 1 \\ \cdot\ u = (\phi_1, \dots, \phi_N) \quad \phi_i \text{ meromorphic section of } K_C^i \\ \qquad\qquad\qquad\qquad\quad \text{(generic)} \\ \cdot\ \vartheta \in \mathbb{R}/2\pi\mathbb{Z} \end{array}\right] \rightsquigarrow \left[\begin{array}{l} \cdot \text{ spectral cover } \quad \Sigma_u \\ \qquad\qquad\qquad\quad \downarrow \pi \qquad \text{degree } N \\ \qquad\qquad\quad \Delta \subset C \\ \qquad\qquad\quad \uparrow \\ \qquad\qquad \text{ramification locus (simple)} \\ \cdot \text{ spectral network } W(u,\vartheta) \text{ on } C \end{array}\right]$$

(show some computer examples)

$W(u,\vartheta)$ jumps at those (u,ϑ) where a BPS web appears:



chamber decomposition: $W(u,\vartheta)$ "constant" in each chamber - walls maybe dense

Def The ij-lift of a path p is the 1-chain $p^{(j)} - p^{(i)} \in C_1(\Sigma)$ where $p^{(i)}$ is the lift of p to sheet Σ.

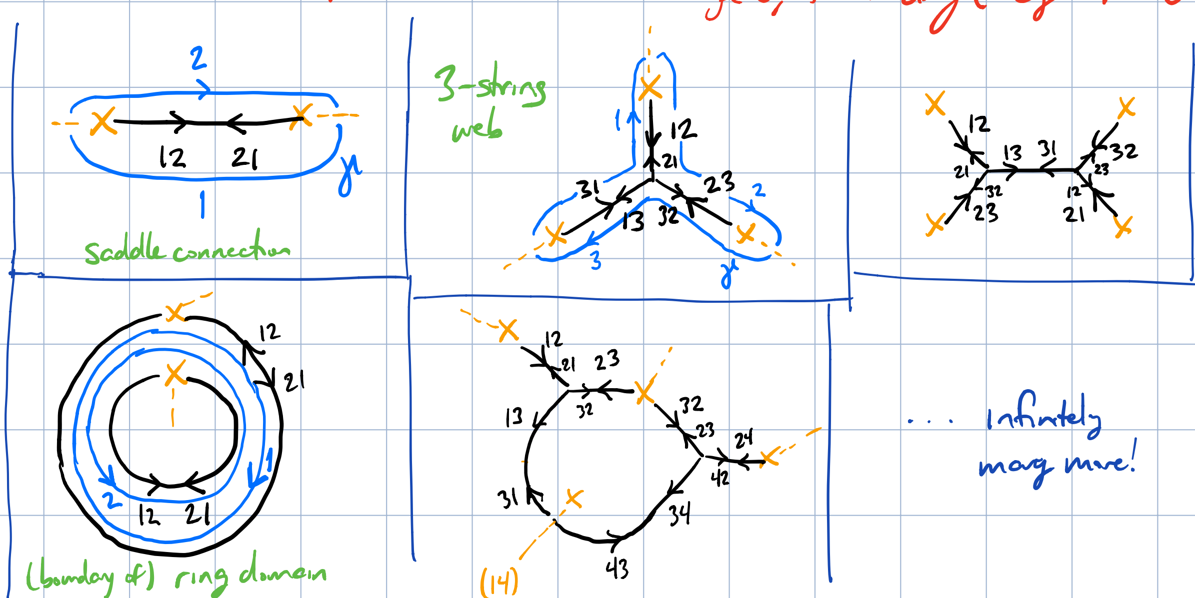
Def A (u,ϑ)-BPS web is a finite set of arcs $p_1, \dots, p_n$ on C, such that

- each p_m lies in a leaf of some $F^{(ij)}(u,\vartheta)$
- the sum of the ij-lifts of the p_m is a 1-cycle on Σ.

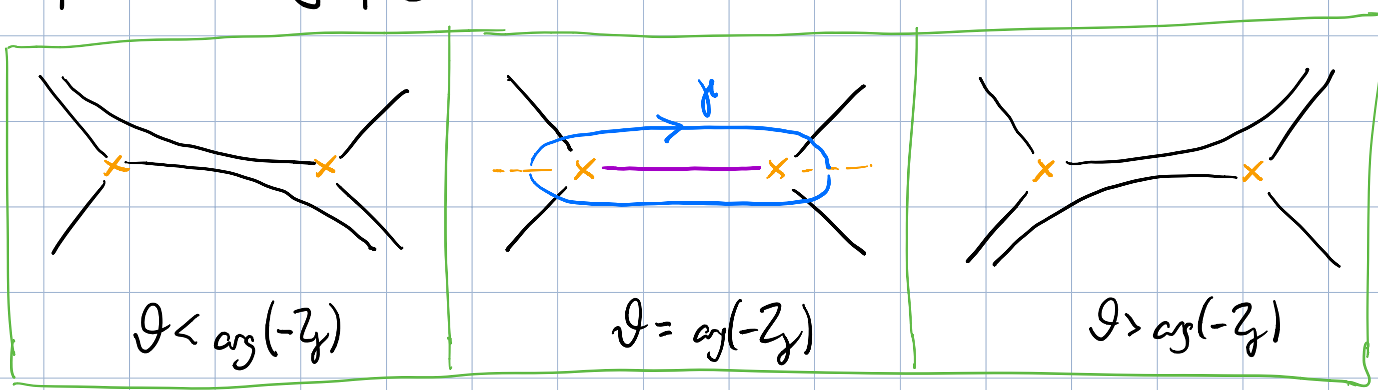
The homology class of this cycle is called the charge of the web.

Fact If there exists a (u,ϑ)-BPS web with charge γ, then $\arg(-Z_\gamma(u)) = \vartheta$.

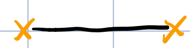
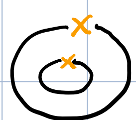
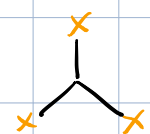
eg:



Basic example of $W(u,\vartheta)$ jumping:



We will <u>quantify</u> the jump of $W(u,\vartheta)$ created by BPS webs into <u>DT invariants</u> $\Omega(\gamma,u) \in \mathbb{Z}$.

picture			
contrib to $\Omega(\gamma)$	$+1$	-2	$+1$

... These contributions will follow from the <u>path lifting</u> rule.

<u>Motivation for path lifting</u>

Say we want to study <u>local systems</u> on C.

Especially, <u>1-parameter families</u> of local systems, obtained from equations

$\nabla(t)\psi = 0$ where $\nabla(t)$ is a flat $GL(N,\mathbb{C})$-connection of the form

Higgs field

$$\nabla(t) = \frac{\varphi}{t} + \cdots$$

Say we fix a loop P on C, let $F_P(t) = \mathrm{Tr}\left(\mathrm{Hol}_{\nabla(t)} P\right)$ $\quad F_P : \mathbb{C}^\times \to \mathbb{C}$

<u>Slogan</u>: a function can be determined knowing only its analytic properties.

So, we want to understand $F_P(t)$ as $t \to 0$.

When $N=1$, simple answer: $F_p(t) \underset{t\to 0}{\sim} \exp\left(Z_p/t\right)$ where $Z_p = \oint_p \varphi$
$\varphi \in \Omega^1(C)$

When $N>1$, first hope would be to just diagonalize φ, let $\Sigma = \{\det(\varphi - \lambda) = 0\} \subset T^*C$,
then $F_p \sim \exp(Z_\gamma/t)$ for $\gamma \in H_1(\Sigma, \mathbb{Z})$. But <u>which</u> γ? Note P might not lift!

Picture we'll find:

$$F_p(t) = \sum_{\gamma \in H_1(\Sigma,\mathbb{Z})} \underbrace{\overline{\Omega}(P,\gamma,u,\vartheta)}_{\in \mathbb{Z}} \mathcal{X}^\vartheta_\gamma(t)$$

and $\mathcal{X}^\vartheta_\gamma(t) \sim \exp\left(Z_\gamma/t\right)$ as $t \to 0$ with $|\arg(t) - \vartheta| < \pi/2$.