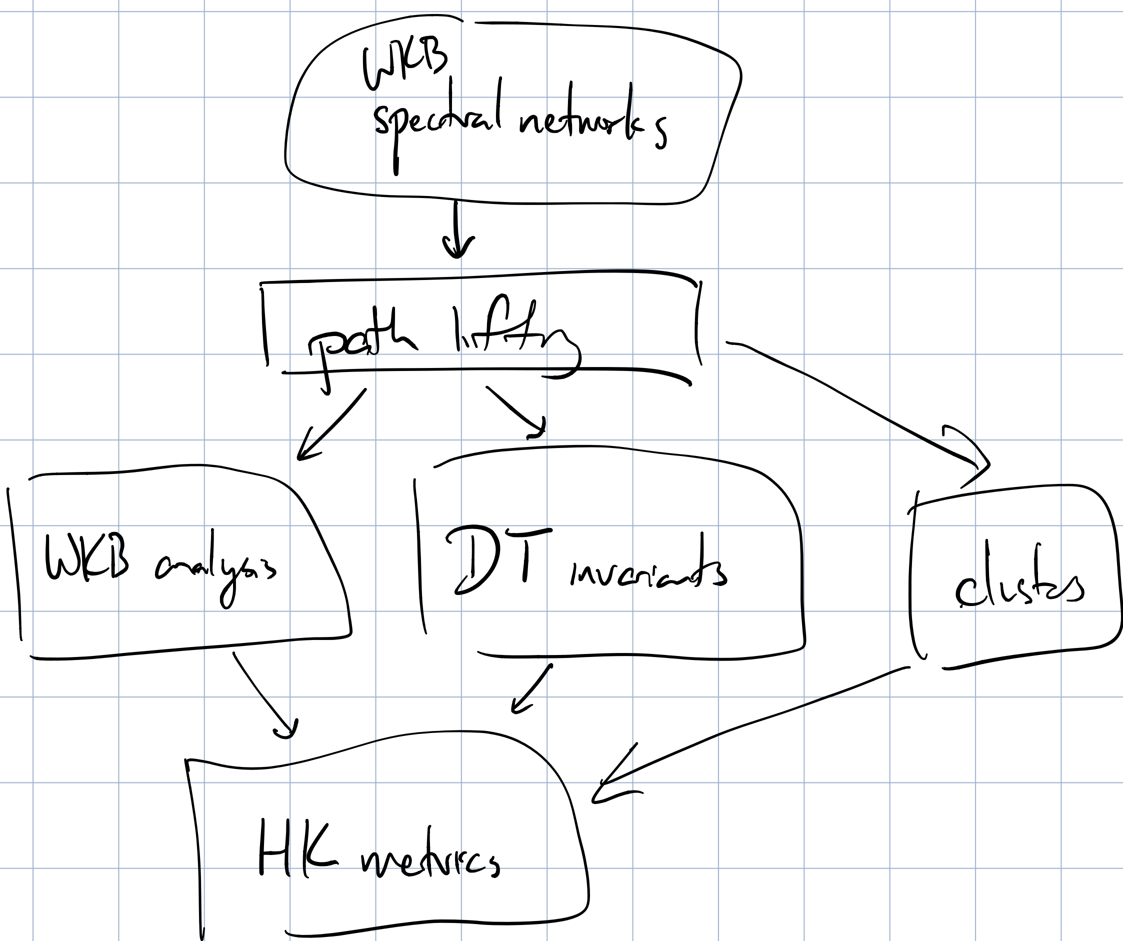
Today: What is a (WKB) spectral network?

Generalization of critical graph of a quadratic differential

Rough structure:

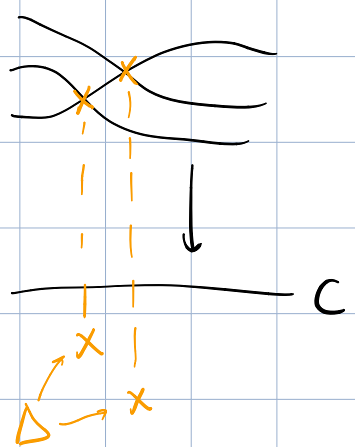


① **Data:**

- Riemann surface C, compact
- $N \geq 1$ $\quad [\mathfrak{g} = \mathfrak{gl}(N)]$
- tuple $(\phi_1, \phi_2, \dots, \phi_N) = u$

  ϕ_i meromorphic section of K_C^i, suff. generic (see ②)

② Define spectral curve $\Sigma_u = \left\{ \sum_{i=0}^{N} \phi_i \lambda^{N-i} = 0 \right\} \subset T^*C$



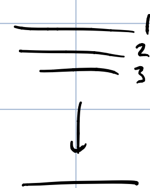
Pick u so that Σ_u has only simple ramification, over $\Delta \subset C$

(later we'll have more restrictions)

③ Local foliations

In simply conn. domain $U \subset C \setminus \Delta$, trivialize

$$\begin{array}{ccc} \Sigma & \supset & \Sigma|_U \\ & & \downarrow \\ C & \supset & U \end{array}$$

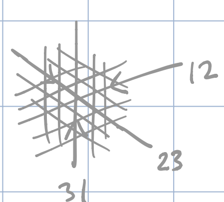


$\rightsquigarrow$ 1-forms $\lambda^{(1)}, \dots, \lambda^{(N)}$ on U

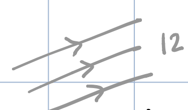
For $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ define oriented foliation $F_\vartheta^{(ij)}$ on U:

leaves are paths with $e^{-i\vartheta}(\lambda^{(i)} - \lambda^{(j)})$ real, negative

eg $N=3$:



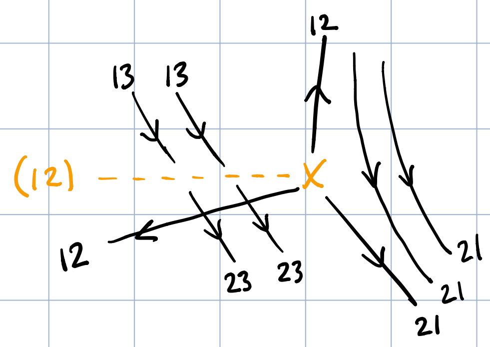
$N=2$:



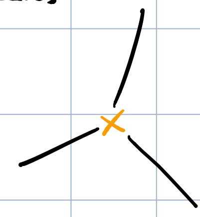
when $\phi_1 = 0$, these are horizontal trajectories of $e^{-2i\vartheta}\phi_2$

④ Global structure

- Under monodromy the foliations $F_\vartheta^{(ij)}$ mix.
- For concrete pictures, trivialize Σ_u away from branch cuts:



- At a branch point $z \in \Delta$ there are 3 singular leaves



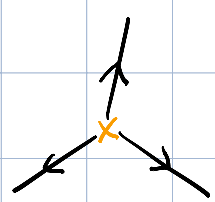
⑤ WKB spectral network

Define $Z_\gamma := \oint_\gamma \lambda \qquad \gamma \in \Gamma = H_1(\Sigma, \mathbb{Z})$

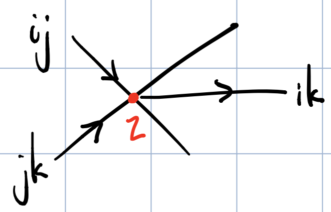
Assume that $\gamma \neq 0 \Rightarrow e^{-i\vartheta} Z_\gamma \notin \mathbb{R}. \qquad (\star)$

Def $\mathcal{W}(\vartheta, u)$ is the smallest collection of trajectories on C s.t.

1) $\mathcal{W}(\vartheta, u)$ contains the 3 critical leaves emanating from each $p \in \Delta$



2) If $\mathcal{W}(\vartheta, u)$ contains an ij-traj and a jk-traj intersecting at z,



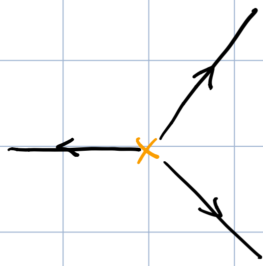
then it also contains an ik-trajectory beginning at z. $\quad$ [Rk $z \notin \Delta$]

Ex 1) $C = \mathbb{CP}^1$

$N = 2$

$u = (0, z\,dz^2)$

$W(\vartheta = 0, u)$:

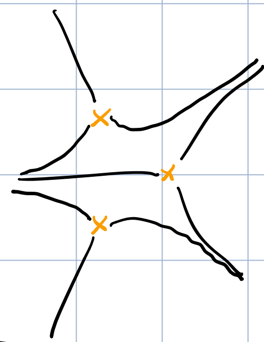


2) $C = \mathbb{CP}^1$

$N = 2$

$u = (0, (z^3-1)\,dz^2)$

$W(\vartheta = 0, u)$:



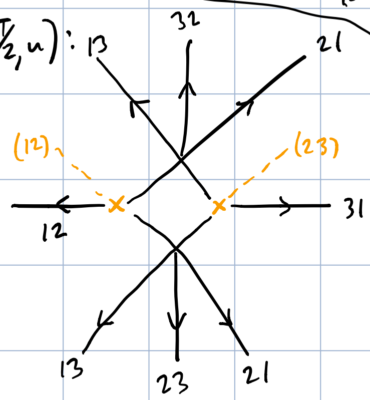
3) $C = \mathbb{CP}^1$

$N = 3$

$u = (0, -dz^2, z\,dz^3)$

$[\Delta = 4z^2 - 27]$

$W(\vartheta = \pi/2, u)$:



(more examples on computer)

Rk
- When ϕ_i are holomorphic, $W(\vartheta, u)$ is "usually" dense on C.
  ——— have poles, ——— sometimes ———.

- The condition $(\star)$ ensures that no ij-trajectory in $W(\vartheta, u)$ runs into an ij-branch pt in Δ.

- Think of $W(\vartheta, u)$ as filtered by length
  "longer paths are less important". Define cutoff version $W(\vartheta; u; \Lambda)$
  by truncating p when $\left|\int \lambda\right|$ reaches Λ. Then $W(\vartheta; u; \Lambda)$ is not dense for any Λ.