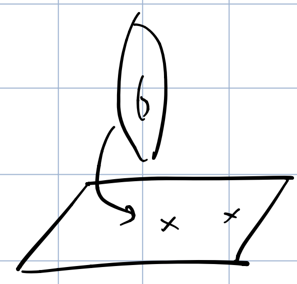
Precision studies of nonabelian Hodge (or: who's afraid of nonabelian Hodge?)

w/ D. Dumas, cf w/ L. Fredrickson, w/ Gaiotto–Moore

① Recollection on nonabelian Hodge

C compact Riemann surface
$G = SU(K)$



Higgs bundle: $(\mathcal{E}, \varphi)$ $\quad$ $\mathcal{E}$ hol v.b., rank K, over C
$\varphi \in H^0(\mathrm{End}\, \mathcal{E} \otimes T^*C)$ $\qquad \mathcal{M}_{Higgs}$

Given a (polystable) Higgs bundle, $\exists$ a harmonic metric h in $\mathcal{E}$ s.t. $F_{D_h} + [\varphi, \varphi^{\dagger_h}] = 0$

(D_h = Chern connection in $\mathcal{E}$, unitary) $\quad$ [Hitchin, Simpson]

Then for any $\zeta \in \mathbb{C}^\times$, $\nabla^\zeta = \frac{\varphi}{\zeta} + D_h + \varphi^{\dagger_h}\zeta$ is a flat complex connection

For any fixed $\zeta \in \mathbb{C}^\times$, the map $(\mathcal{E}, \varphi) \longmapsto \nabla^\zeta$ induces a diffeomorphism of moduli spaces,

$\mathrm{NAH}^\zeta : \mathcal{M}_{Higgs} \xrightarrow[\text{diffeo}]{\sim} \mathcal{M}_{flat}$ $\quad$ "nonabelian Hodge map" (NAH)

[The NAH^ζ induce a hyperkähler structure on $\mathcal{M}_{Higgs}$.
In particular, a metric, whose Kähler form is $\omega = \mathrm{Re}\left(\Phi^{1*}\Omega_{ABG}\right)$
cf. Zhang talk
Q: What can we understand explicitly?]

② Some baby examples

$\exists$ an analogue of this theory where φ is allowed poles, subject to some conditions.

Fix N, then $\exists$ a space $\mathcal{M}_{K,N}$ of Higgs bundles over $\mathbb{CP}^1$, where eigenvalues of φ have

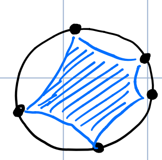
$\lambda \sim z^{N/K} dz$. $\quad$ [w/ Laura Fredrickson]

Inside $\mathcal{M}_{K,N}$ there's a half-dimensional slice, "Hitchin component"

e.g. $K=2$: $\left\{ \mathcal{E} = \mathcal{O}(\tfrac{N}{4}) \oplus \mathcal{O}(-\tfrac{N}{4}),\ \varphi = \begin{pmatrix} 0 & 1 \\ P_2(z) & 0 \end{pmatrix} dz \text{ with } P_2(z) = z^N + (\text{degree} \leq \tfrac{N}{2} - 1) \right\}$ $\quad$ coeff of P_2

$\Big\updownarrow$ NAH at $\zeta = 1$ $\qquad$ $\Big\updownarrow$ diffeo $\dim = 2N-4$

$SL_2\mathbb{R} \backslash \left\{ \text{harmonic maps } F : \mathbb{C} \longrightarrow SL_2\mathbb{R}/SO_2 \text{ with image ideal } (N+2)\text{-gon} \right\}$ $\quad$ cross ratios of $N+2$ pts in $\mathbb{RP}^1$



[Han–Tam–Treibergs–Wan] [Wolf] $\quad$ (Q. Li talk) e.g. $K=3$: $\left\{ \mathcal{E} = \mathcal{O}(\tfrac{N}{3}) \oplus \mathcal{O} \oplus \mathcal{O}(-\tfrac{N}{3}),\ \varphi = \begin{pmatrix} 0 & 1 & 0 \\ P_2(z) & 0 & 1 \\ P_3(z) & P_2(z) & 0 \end{pmatrix} dz \text{ with } \begin{array}{l} P_2(z) = (\text{degree} \le \tfrac{N}{3}-1) \\ P_3(z) = z^N + (\text{degree} \le \tfrac{2N}{3}-1) \end{array} \right\}$ $\boxed{\text{coeff of } P_2, P_3}$

$\updownarrow$ NAH at $\zeta = 1$ $\qquad\qquad$ $\updownarrow$ differ.

$\left\{ \text{harmonic maps } F: \mathbb{C} \longrightarrow SL_3\mathbb{R}/SO_3 \text{ with "polynomial growth"} \right\} \big/ SL_3\mathbb{R}$ $\qquad$ $\boxed{\text{cross ratios of } N+3 \text{ pts in } \mathbb{RP}^2 \text{ (convex polygon)}}$

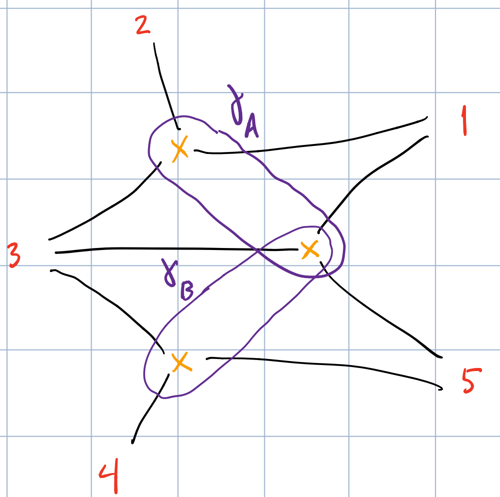
[Loftin] [Labourie] [Dumas-Wolf] when $P_2 = 0$: polynomial affine spheres

③ Some predictions [Gaiotto-Moore-N, N]

Take $K=2$. Suppose we want to compute the polygon corresp. to $P_2 = z^3 - 1$. (or nearly)

Data we need:

a) critical graph of $P_2\, dz^2$

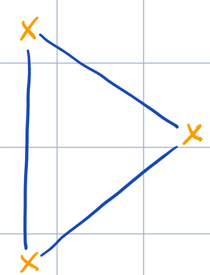


b) periods $Z_\gamma = \oint_\gamma \sqrt{P_2}\, dz$ $\qquad \gamma \in H_1(\Sigma, \mathbb{Z}) = \Gamma$

$\Sigma = \{ y^2 + P(z) = 0 \}$

$Z_{A/B} = -2.524\ldots \pm 1.457\ldots i$

c) list of all saddle connections of $e^{i\vartheta} P_2\, dz^2$ — encoded in $\Omega: \Gamma \to \mathbb{Z}$



$\Omega(\pm\gamma_A) = 1$

$\Omega(\pm\gamma_B) = 1$

$\Omega(\pm(\gamma_A + \gamma_B)) = 1$

Let $\mathcal{X}_A, \mathcal{X}_B$ be cross ratios of the polygon: $\mathcal{X}_A = r_{1235}$ $\quad \mathcal{X}_B = r_{1345}$ $\qquad (\mathcal{X}_A = \mathcal{X}_B)$

Weakest prediction: if we replace $P_2 \to R^2 P_2$ then $\mathcal{X}_A = \exp\left(-5.048\ldots R + \delta(R)\right)$, $\lim_{R\to\infty} \delta(R) = 0$

$$= \exp\left(2R\, \mathrm{Re}(Z_A) + \delta(R)\right)$$

Weak prediction: $\delta(R) = -\dfrac{3}{2\sqrt{\pi R |Z|}}\, e^{-2|Z|R} + \tilde{\delta}(R)$ $\qquad \lim_{R\to\infty} \sqrt{R}\, e^{2R|Z_{min}|}\, \tilde{\delta}(R) = 0$

$$= \sum_{\mu \in \Gamma} \frac{\Omega(\mu) \langle \gamma_A, \mu \rangle}{4\pi i} \frac{\alpha_\mu + 1}{\alpha_\mu - 1} \sqrt{\frac{\pi}{R|Z_\mu|}}\, e^{-2|Z_\mu|R} + \tilde{\delta}(R)$$

Strong prediction: $\mathcal{X}_A$ can be extended to a piecewise analytic function $\mathcal{X}_A(\zeta)$, $\zeta \in \mathbb{C}^\times$, obeying the integral eqⁿ

$$\mathcal{X}_A(\zeta) = \exp\left[\left(\zeta^{-1} Z_A + \zeta \bar{Z}_A\right) + \frac{1}{4\pi i} \sum_{\mu \in \Gamma} \Omega(\mu) \langle \gamma_A, \mu \rangle \int_{\zeta' \in Z_A \mathbb{R}_-} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left(1 + \mathcal{X}_\mu(\zeta')\right) \right]$$

[Why? Study abelianization of the family of flat connections ∇^ζ and their asymptotics as $\zeta \to 0$; "enjoys" Stokes phenomena controlled by the saddle connections; int eq. gives simplest realiz^n]

$\exists$ similar predictions for some $K=3$ examples eg $P_3 = z^3 - 3z - 2$

(4) Numerical results

[Dumas-Wolf] code for computing the polygons in $K=3$ case, by directly solving PDE.

[Dumas-N] —— $K=2,3$ —— or integral equations.

e.g. for $K=2$, have to solve $\Delta u - 4e^{2u} + 4e^{-2u}|P_2|^2 = 0$

Show results. Large R: best for $\int$ eq.
Small R: best for direct PDE.

(show example $R=2$)

(5) Hyperkähler metrics

$\Omega = d\log \mathcal{X}_A \wedge d\log \mathcal{X}_B \Rightarrow$ use these formulas to predict the HK metric

eg "weakest prediction" $\Rightarrow$ HK metric has $\|\dot{P}_2\|^2 = \int_{\mathbb{C}} \frac{|\dot{P}_2|^2}{|P_2|}$

"semiflat metric" cf. [Mazzeo-Swoboda-Weiss-Witt]

Show results.