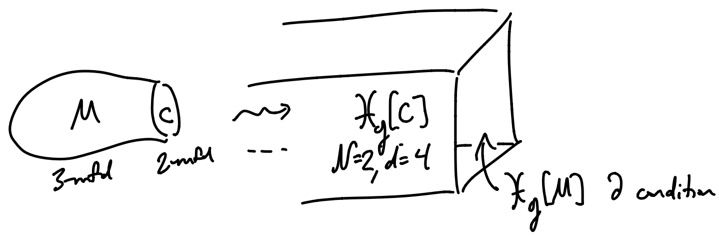
WIP w/ D. Freed.

Physics context: M 3-manifold $\rightsquigarrow \mathcal{X}_{\mathfrak{g}}[M]$ $\mathcal{N}=2$, $d=3$ SCFT [DGG, Yamazaki, ...]
$\mathfrak{g}$ of ADE type



Try to understand $\mathcal{X}_{\mathfrak{g}}[M]$ by studying its vacuum structure and UV $\to$ IR maps for operators of various dimensions.

Vacua of $\mathcal{X}_{\mathfrak{g}}[M \times S^1] \longleftrightarrow$ flat $G_{\mathbb{C}}$-connections ∇ on M

$\widetilde{W}_{eff}$(vacuum) $\longleftrightarrow \log CS(\nabla)$

What I'll describe: an "IR" picture of the flat $G_{\mathbb{C}}$-connections and $CS(\nabla)$, when $G_{\mathbb{C}} = GL_K\mathbb{C}$
(Morally: wrap K branes on $M \subset T^*M$, then deform to 1 brane on $\widetilde{M} \subset T^*M$)

Main picture:

$G_{\mathbb{C}} = GL_K\mathbb{C}$

$\widetilde{M}, \widetilde{\nabla}$ flat $GL_1\mathbb{C}$-conn with "hyper" singularities

π branched $K:1$ cover $\downarrow$

M, ∇ flat $GL_K\mathbb{C}$-conn

$\nabla, \widetilde{\nabla}$ related by "abelianization"

$$\Rightarrow \widehat{CS}(\widetilde{M}, \widetilde{\nabla}) = CS(M, \nabla)$$

$\uparrow$ deformed $GL(1,\mathbb{C})$ CS $\qquad \uparrow$ ordinary $GL(K,\mathbb{C})$ CS

[cf. CCV]

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① Classical Chern-Simons

$\left.\begin{array}{l} M \text{ compact spin 3-mfd} \\ \nabla \text{ flat } GL_K\mathbb{C}\text{-conn.} \end{array}\right] \rightsquigarrow CS(M,\nabla) \in \mathbb{C}^\times$

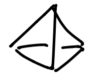
If $\nabla = d + A$, $A \in \Omega^1(M, \mathfrak{g})$ then $CS(M,\nabla) = \exp\left[\frac{1}{4\pi i}\int_M Tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)\right]$

Similarly for $G = PSL_K\mathbb{C}$ or (M with boundary, ∇ boundary-unipotent)

Facts:

a) $\{CS(M,\nabla)\} \subset \mathbb{C}^\times$ is a top. inv't of M.

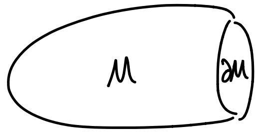
b) M hyperbolic $\Rightarrow CS(M, \nabla^{hyp}) = \exp\left(-\frac{vol(M)}{2\pi} + i(\cdots)\right)$

c) M triangulated $\Rightarrow$ some ∇ can be described by $\frac{1}{6}(K^3-K)$ param. $\mathcal{X}_i \in \mathbb{C}^\times$ for each 

$$CS(M,\nabla) = \exp\left(\frac{1}{2\pi i}\sum_i Li_2(\mathcal{X}_i)\right)$$

[Thurston, Dupont, .., DGG, GGTZ]

d) $CS(M,\nabla)$ fits into an invertible TFT:



$CS(M,\nabla) \in CS(\partial M, \nabla|_{\partial M})$

M	$CS(M,\nabla)$
3-mfd	number $\in \mathbb{C}^\times$
2-mfd	vector space 1-dim
$\vdots$	$\vdots$

ie a functor $Bord_{GL_K} \longrightarrow Lines$

"manifolds with flat GL_K-conn"

e) if we don't require ∇ flat, then ∇ flat $\Longleftrightarrow$ ∇ is crit pt for $CS(M,\nabla)$

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② Defomed Chern-Simons

Now take $\widetilde{M}$ with $GL_1\mathbb{C}$-conn $\widetilde{\nabla}$.

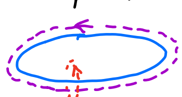
Idea: fix codim-2 "hypers" $H_i \subset \widetilde{M}$, let $\mathcal{X}_i = hol_{H_i}\widetilde{\nabla} \in \mathbb{C}^\times$

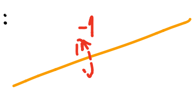
$$\widetilde{CS(\widetilde{M},\widetilde{\nabla})} = \exp\left[\frac{1}{4\pi i}\int A\wedge dA + \frac{1}{2\pi i}\sum Li_2(\mathcal{X}_i)\right]$$

then $\widetilde{CS(\widetilde{M},\widetilde{\nabla})}$ extremized $\Rightarrow$ $\boxed{F_\nabla = -\sum_i \log(1-\mathcal{X}_i)\,\delta_{H_i}}$

Deformed CS TFT:

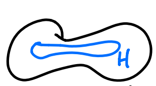
new category $\widetilde{Bord}_{GL_1}$: spin manifolds $\widetilde{M}$, $\widetilde{\nabla}$ flat $GL_1\mathbb{C}$-conn over $\widetilde{M}$ with defects

Ⓐ hypers: 

Ⓑ branch: 

(+ extra data)

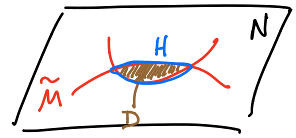
Have a TFT $\widetilde{Bord}_{GL_1} \xrightarrow{\widetilde{CS}} Lines$

What is $\widetilde{CS}($  $)$?

Use ordinary CS on $M\setminus H$, glue in an object "$\exp\left(\frac{1}{2\pi i}Li_2(\mathcal{X})\right)$" $\in CS(T^2, \widetilde{\nabla}|_{T^2})$ around H.

Rk This kind of deformation of CS appears in topological A model

$\tilde{M} \subset N$ $\rightsquigarrow$ CS on $\tilde{M}$ defined by holomorphic discs D
Lag symp
with $\partial D = H$
isolated disc contributes $\frac{1}{2\pi i} \mathrm{Li}_2(\mathcal{X})$



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③ Abelianization

Thm (in prog)

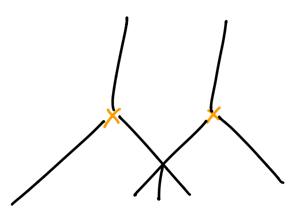
$$\begin{tikzcd} & \widetilde{\mathrm{Bord}}_{GL_1} \arrow[dr] & \\ \mathrm{Bord}_{Ab} \arrow[ur] \arrow[dr] & & \mathrm{Lines} \\ & \mathrm{Bord}_{GL_K} \arrow[ur] & \end{tikzcd}$$

commutes

where Bord_{Ab} is new category of "abelianizations": tuples $(M, \nabla, \tilde{M}, \tilde{\nabla}, \pi, \iota)$

- $(M, \nabla) \in \mathrm{Bord}_{GL_K}$
- $(\tilde{M}, \tilde{\nabla}) \in \mathrm{Bord}_{GL_1}$
- $\pi : \tilde{M} \to M$ branched $K:1$
- $\iota : \pi_* \tilde{\nabla} \to \nabla$ piecewise flat, jumps at codim-1 "spectral network" $\subset M$ by unipotents $S \in \mathrm{Aut}(\pi_* \tilde{\nabla})$
- other conditions near defects

Typical shape of walls:



$\times$ = branch locus of π

ie: ∇ and $\pi_* \tilde{\nabla}$ differ by "cutting + gluing" along this picture.

Topological examples

If M is a Δ'd 3-mfd, $K=2$ then some ∇ can be lifted to Bord_{Ab} and then computing $\widehat{CS}(\tilde{M}, \tilde{\nabla})$ "explains" the dilog formula
(probably also for $K>2$ but not proved yet)

Physical examples

- If C is a Riem. surf. then every $u \in$ Coulomb branch of $\mathfrak{X}_{\mathfrak{g}}[C]$ $(C=M)$
  determines a spectral network, almost every ∇ lifts to Bord_{Ab}
  (used for UV $\to$ IR map in $\mathfrak{X}_{\mathfrak{g}}[C]$ [GMN])
  Walls of spectral network $\longleftrightarrow$ BPS solitons on canonical surface defect in $\mathfrak{X}_{\mathfrak{g}}[C]$.
  $\longleftrightarrow$ holomorphic bigons in T^*C.

- If $\widetilde{M} \subset T^*M$ can try to determine a spectral network using hol bigons again.
  View passage to $\widetilde{M}$ as a mass deformation of $\mathfrak{X}_{\mathfrak{g}}[M]$.
  Lift to Bord_{Ab} describes UV $\to$ IR map.
  But, some vacua don't have such lifts (for fixed $\widetilde{M}$ and network).
  What happened to them?