Comparing signs in wall-crossing formulas

Andrew Neitzke

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In this note I describe some sign issues that come up in the study of the wall-crossing automorphisms introduced by Kontsevich-Soibelman. The reason for writing this is that these signs are constantly causing confusion and I want to have somewhere to point when it comes up, rather than just saying "well, the signs are complicated."

Automorphisms (twisted). Fix a lattice Γ with antisymmetric bilinear pairing $\langle\cdot,\cdot\rangle$. Kontsevich-Soibelman [3] consider the *twisted torus* A, a field generated over (say) $\mathbb{C}$ by indeterminates $\{X_\gamma\}_{\gamma\in\Gamma}$ subject to the relation

$$X_\gamma X_\mu = (-1)^{\langle\gamma,\mu\rangle} X_{\gamma+\mu}. \tag{0.1}$$

For each $\gamma \in \Gamma$ they also introduce an automorphism $\mathcal{K}_\gamma$ of A, given by

$$\mathcal{K}_\gamma(X_\mu) = X_\mu(1 - X_\gamma)^{\langle\gamma,\mu\rangle}. \tag{0.2}$$

They then write various identities in certain formal completions of groups of automorphisms of A, such as

$$\mathcal{K}_\gamma\mathcal{K}_\mu = \mathcal{K}_\mu\mathcal{K}_{\mu+\gamma}\mathcal{K}_\gamma \qquad \text{if } \langle\gamma,\mu\rangle = 1, \tag{0.3}$$

$$\mathcal{K}_\gamma\mathcal{K}_\mu = \left(\prod_{n=0}^{\infty}\mathcal{K}_{\mu+n(\gamma+\mu)}\right)\mathcal{K}_{\mu+\gamma}^{-2}\left(\prod_{n=\infty}^{0}\mathcal{K}_{\gamma+n(\gamma+\mu)}\right) \qquad \text{if } \langle\gamma,\mu\rangle = 2. \tag{0.4}$$

Automorphisms (untwisted). Now suppose we insist on working with an *untwisted* torus $\tilde{A}$, generated by monomials $\{\tilde{X}_\gamma\}_{\gamma\in\Gamma}$, with the multiplicative law

$$\tilde{X}_\gamma\tilde{X}_\mu = \tilde{X}_{\gamma+\mu}. \tag{0.5}$$

The two fields A and $\tilde{A}$ are isomorphic, but not canonically so. To get an isomorphism between the two, we need to choose a *quadratic refinement* of the pairing $(-1)^{\langle\cdot,\cdot\rangle}$, i.e. a map

$$\sigma : \Gamma \to \{\pm 1\} \tag{0.6}$$

which fails to be a homomorphism, in a controlled way,

$$\sigma(\gamma)\sigma(\mu) = (-1)^{\langle\gamma,\mu\rangle}\sigma(\gamma+\mu). \tag{0.7}$$

Having chosen σ, we can identify A and $\tilde{A}$ by mapping

$$X_\gamma \mapsto \sigma(\gamma)\tilde{X}_\gamma. \tag{0.8}$$

By conjugation with this isomorphism, we can also transfer the automorphisms $\mathcal{K}_\gamma$ of A to automorphisms $\tilde{\mathcal{K}}_\gamma$ of $\tilde{A}$, which however depend on σ:

$$\tilde{\mathcal{K}}_\gamma(\tilde{X}_\mu) = \tilde{X}_\mu(1-\sigma(\gamma)\tilde{X}_\gamma)^{\langle\gamma,\mu\rangle}. \tag{0.9}$$

In particular, the identities (0.3), (0.4) are still true if we replace $\mathcal{K}$ by $\tilde{\mathcal{K}}$ everywhere.

If we are in some situation where we have a canonical choice of σ, then we can freely use either (0.1) or (0.9) as we like.

Generating functions. An automorphism of the form

$$\mathcal{S} = \prod_{n=1}^{\infty} \mathcal{K}_{n\gamma}^{\Omega(n\gamma)} \tag{0.10}$$

(for some fixed γ) can be written as

$$\mathcal{S}(X_\mu) = f(X_\gamma)^{\langle\gamma,\mu\rangle} X_\mu \tag{0.11}$$

where we introduced the generating function

$$f(X_\gamma) = \prod_{n=1}^{\infty}(1-X_{n\gamma})^{\Omega(n\gamma)}. \tag{0.12}$$

If we have a quadratic refinement σ, we can transfer to the untwisted world: an automorphism of the form

$$\tilde{\mathcal{S}} = \prod_{n=1}^{\infty} \tilde{\mathcal{K}}_{n\gamma}^{\Omega(n\gamma)} \tag{0.13}$$

(for some fixed γ) can be written as

$$\tilde{\mathcal{S}}(\tilde{X}_\mu) = \tilde{f}(\tilde{X}_\gamma)^{\langle\gamma,\mu\rangle} \tilde{X}_\mu \tag{0.14}$$

where

$$\tilde{f}(\tilde{X}_\gamma) = \prod_{n=1}^{\infty}(1-\sigma(n\gamma)\tilde{X}_{n\gamma})^{\Omega(n\gamma)}. \tag{0.15}$$

Example. Now suppose we consider a lattice Γ of rank 2, with generators γ and μ. There is a quadratic refinement with $\sigma(\gamma) = -1$ and $\sigma(\mu) = -1$. Suppose moreover that $\langle\gamma,\mu\rangle = 2$. Then from (0.7) it follows that

$$\sigma(\gamma+n(\gamma+\mu)) = -1, \qquad \sigma(\mu+\gamma) = +1, \qquad \sigma(\mu+n(\gamma+\mu)) = -1. \tag{0.16}$$

The corresponding generating functions $\tilde{f}$ for the automorphisms appearing on the right side of (0.4), after passing to the untwisted world using σ, would thus be of the form

$$(1+X_{\gamma+n(\gamma+\mu)}), \qquad (1-X_{\gamma+\mu})^{-2}, \qquad (1+X_{\mu+n(\gamma+\mu)}). \tag{0.17}$$

This matches with the functions appearing in [2] Example 1.6, case $\ell_1 = \ell_2 = 2$. (Actually, because of a slightly different convention, the functions appearing there are the square of the functions we have here; what I am trying to emphasize is that the signs match.) So it seems that the correct way of comparing [2] and [3] is to use this particular σ.

Note that after this untwisting the function $\tilde{f}$ has a positivity property: if it is expanded in powers of the formal variable appearing in its argument, all coefficients are positive (we use $1/(1-x) = 1 + x + x^2 + \cdots$)

Example. Suppose we consider again a lattice Γ of rank 2, with generators γ and μ. Suppose that $\langle \gamma, \mu \rangle = m$. Now consider expanding the automorphism $\mathcal{K}_\gamma \mathcal{K}_\mu$ as we did in (0.3), (0.4) above: we will get a huge product of the form $\prod_{p,q \geq 0} \mathcal{K}_{p\gamma+q\mu}^{\Omega(p\gamma+q\mu)}$, taken in order of increasing p/q. In particular, the factors with $p/q = 1$ make up an automorphism $\mathcal{S}$ with corresponding generating function $f(X_{\mu+\gamma})$. We will study just this automorphism.

There is a quadratic refinement with $\sigma(\gamma) = -1$ and $\sigma(\mu) = -1$. Then we will have

$$\sigma(n(\gamma+\mu)) = (-1)^{mn} = \begin{cases} (-1)^n & \text{if } m \text{ odd,} \\ +1 & \text{if } m \text{ even.} \end{cases} \tag{0.18}$$

Use this quadratic refinement to pass from f to $\tilde{f}$. Experimentally it seems that the numbers $\Omega(n(\gamma+\mu))$ which go into f are all negative if m is even, and have the sign of $(-1)^n$ if m is odd (see e.g. Table C.9 on page 97 of [1].) Using this together with the explicit form of σ above we find that $\tilde{f}(\tilde{X}_{\mu+\gamma})$ involves only positive powers of $(1 + \tilde{X}_{\mu+\gamma})$ and only negative powers of $(1 - \tilde{X}_{\mu+\gamma})$, so that once again, if it is expanded in powers of $\tilde{X}_{\mu+\gamma}$ all coefficients are positive.

References

1. D. Galakhov, P. Longhi, T. Mainiero, G. W. Moore, and A. Neitzke (2013). Wild Wall Crossing and BPS Giants. *JHEP* **2013**(11). arXiv:`1305.5454 [hep-th]`.
2. M. Gross, R. Pandharipande, and B. Siebert (Feb. 2009). The tropical vertex. eprint: `0902.0779`.
3. M. Kontsevich and Y. Soibelman (2008). Stability structures, motivic Donaldson-Thomas invariants and cluster transformations. eprint: `0811.2435`.