Geometry of Stokes Phenomena

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Preliminary and **incomplete** notes

These are notes from a spring 2022 course at Yale University. They are extremely incomplete, unreliable, full of mistakes and omissions, and still being updated. A PDF built from the master branch, which should be up to date, can be found at

https://www.dropbox.com/s/bqot580bfxwr459/stokes-geometry.pdf?dl=0

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1. Intro

1.1. **Introduction.** What is the Stokes phenomenon? I think this question does not have a completely sharp answer, but in this section I make a preliminary attempt to give the right flavor. Perhaps by the end of the course we will be able to do better.

In a first course in complex analysis one learns that holomorphic functions have many beautiful properties, and likewise for meromorphic functions. In contrast one learns relatively little about functions with essential singularities (the principal exception being Picard's theorem, which might serve to convince you that essential singularities are something wild and best avoided). Some sorts of essential singularity have a beautiful structure of their own. The Stokes phenomenon is one aspect of this structure.

1.2. The Airy function.

1.2.1. *Definition.* The classic example studied by Stokes concerns the Airy function,[1]

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}s \, \exp\left(\mathrm{i}\left(\frac{1}{3}s^3 + zs\right)\right). \tag{1.1}$$

Airy showed in 1838 that, for $z < 0$, the function $\mathrm{Ai}(z)$ describes the intensity pattern of light observed near the primary arc in a monochromatic rainbow (for an account see [1].) In particular, the maxima of $\mathrm{Ai}(z)$ tell you where the supernumerary arcs (fringes that appear near the primary arc) will be found. So, to test this theory, it is important to be able to actually calculate $\mathrm{Ai}(z)$.

1.2.2. *Asymptotics along the real line.* Stokes approached this problem by considering the behavior of $\mathrm{Ai}(z)$ as $z \to \infty$ ("expanding around" the essential singularity at $z = \infty$). This can be done by the method of complex steepest descent, reviewed in Appendix A. It involves decomposing $\mathrm{Ai}(z)$ into integrals $\psi^{(i)}(z)$ over more general contours, whose asymptotic behavior is more elementary. After so doing, one finds that the behaviors for $z \in \mathbb{R}_+$ and $z \in \mathbb{R}_-$ are very different. For $z \in \mathbb{R}_+$ there is real exponential decay,[2]

$$\mathrm{Ai}(z) = \psi^{(1)}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \mathrm{e}^{-\frac{2}{3}z^{\frac{3}{2}}} \qquad \text{as } z \to \infty \text{ in } \mathbb{R}_+ . \tag{1.2}$$

[1] This is an example of an *oscillating integral*. It is not absolutely convergent but conditionally convergent, irrespective of the value of z.

[2] For two functions f, g, the asymptotic notation $f \sim g$ means that $\lim \frac{f}{g} = 1$.

For $z \in \mathbb{R}_-$ the behavior is more complicated: there are two complex exponentials which combine to a real oscillatory sum. Precisely, one has a decomposition[3]

$$\mathrm{Ai}(z) = \psi^{(1)}(z) + \mathrm{i}\psi^{(2)}(z), \tag{1.3}$$

$$\psi^{(1)}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \mathrm{e}^{-\frac{2}{3}z^{\frac{3}{2}}} \qquad \text{as } z \to \infty \text{ in } \mathbb{R}_-\,, \tag{1.4}$$

$$\psi^{(2)}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \mathrm{e}^{\frac{2}{3}z^{\frac{3}{2}}} \qquad \text{as } z \to \infty \text{ in } \mathbb{R}_-\,. \tag{1.5}$$

Combining these asymptotics gives an approximate formula

$$\mathrm{Ai}(z) \approx \frac{1}{\sqrt{\pi}} |z|^{-\frac{1}{4}} \sin\left(\frac{2}{3}|z|^{\frac{3}{2}} + \frac{\pi}{4}\right) \qquad \text{as } z \to \infty \text{ in } \mathbb{R}_-\,. \tag{1.6}$$

These asymptotics, although they are only statements about the limit $z \to \infty$, turn out to give remarkably good approximations even for finite z: see Figure 1. (For example, one can use (1.6) to determine the position of the supernumerary arcs, to good accuracy.)

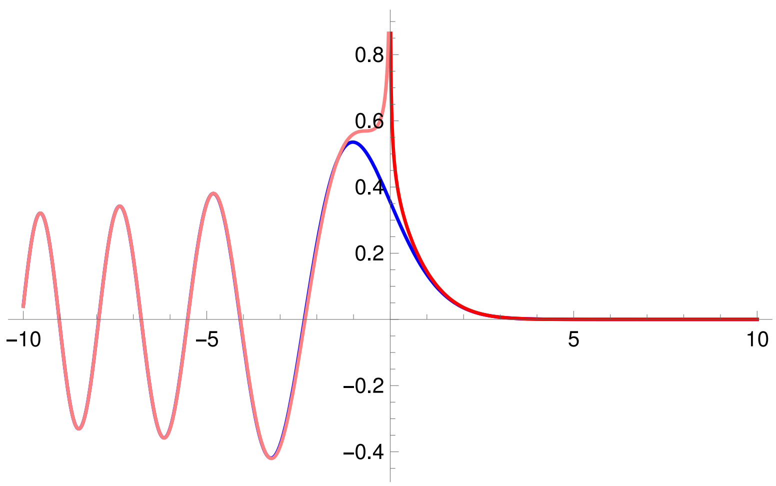

FIGURE 1. Blue: $\mathrm{Ai}(z)$ for real z. Red: the asymptotic (1.2). Pink: the asymptotic (1.3).

Exercise 1.1. Use the method of complex steepest descent to derive the asymptotics (1.2) and (1.3). (In both cases some changes of variable and contour deformations will be required.)

The function $\mathrm{Ai}(z)$ is entire; in particular, by analytic continuation, its values on $\mathbb{R}_-$ are completely determined by its values on $\mathbb{R}_+$. The asymptotics (1.2) also admit analytic continuation, at least along any path avoiding $z = 0$. However, analytic continuation of the asymptotics from (1.2) does not give the asymptotics (1.3): it misses the second term. This is not a contradiction, but it does suggest that something interesting is going on here.

[3]The formulas (1.4) and (1.5) involve various fractional powers of z, which need to be carefully defined. We choose the branches which are reached by analytic continuation in the upper half-plane. In the rest of this section, we will sometimes consider more general complex z, and then our convention is to place the branch cut on the ray $\arg z = \frac{4\pi}{3}$. None of the important structure really depends on where we choose to put the branch cut, but the details of the formulas do.

1.2.3. *Asymptotics along general rays.* To investigate more closely, we examine the behavior of $\mathrm{Ai}(z)$ as $z \to \infty$ in more general directions. For an interval $I \subset \mathbb{R}$ define the sector

$$\mathbb{S}_I = \{z : \arg(z) \in I\} \subset \mathbb{C}. \tag{1.7}$$

Then, again by the complex steepest descent method, one finds that:

- the simple asymptotics (1.2) hold for $z \to \infty$ in $\mathbb{S}_{(-\frac{2\pi}{3},\frac{2\pi}{3})}$,
- the more complicated structure (1.3) holds for $z \to \infty$ in $\mathbb{S}_{(\frac{2\pi}{3},\frac{4\pi}{3})}$.

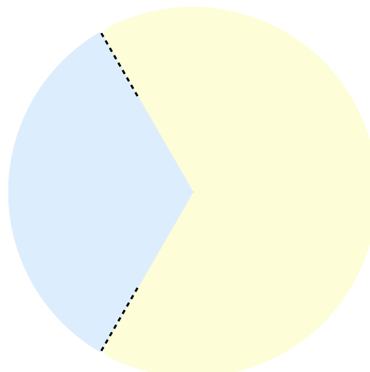

FIGURE 2. In the yellow sector the asymptotics of $\mathrm{Ai}(z)$ are given by (1.2); in the blue sector they are given by (1.3). The boundary rays are Stokes rays (to be defined below).

Exercise 1.2. Derive these statements using the method of complex steepest descent.

We should comment on one tricky point. The function $\mathrm{Ai}(z)$ is entire and in particular it is continuous. We have two different decompositions of this function in the two sectors. In both sectors we have a term $\psi^{(1)}$, but in the blue sector we also have the extra term $\mathrm{i}\psi^{(2)}$. It follows that $\psi^{(1)}$ is only piecewise continuous: it jumps by $\mathrm{i}\psi^{(2)}$ as we cross the boundary ray $\ell_{\frac{2\pi}{3}}$. On the other hand, the asymptotic behavior of $\psi^{(1)}$ as $z \to \infty$ is perfectly continuous in a neighborhood of this ray. How can this be consistent? The resolution is that in the sector $\mathbb{S}_{(\frac{\pi}{3},\pi)}$ we have $\mathrm{Re}\, z^{\frac{3}{2}} < 0$, and thus[4]

$$\psi^{(2)} \ll \psi^{(1)} \qquad \text{as } z \to \infty \text{ in } \mathbb{S}_{(\frac{\pi}{3},\pi)}. \tag{1.8}$$

Thus we have

$$\psi^{(1)} \sim \psi^{(1)} + C\psi^{(2)} \qquad \text{as } z \to \infty \text{ in } \mathbb{S}_{(\frac{\pi}{3},\pi)} \tag{1.9}$$

for any $C \in \mathbb{C}$. Said otherwise, at the moment when the discontinuity of $\psi^{(1)}$ appears, it is too small to be observed in the asymptotic behavior.

At any rate, we see that the behavior of $\mathrm{Ai}(z)$ around $z = \infty$ is quite different from the behavior of a meromorphic function $\phi(z)$ around a pole or a regular point. Indeed, around such a point we would have uniform asymptotics

$$\phi(z) \sim cz^k \qquad \text{as } z \to 0 \text{ in any direction} \tag{1.10}$$

[4]For two functions f, g the notation $f \ll g$ means $\lim f/g = 0$.

for some $k \in \mathbb{Z}$. For $\mathrm{Ai}(z)$ we are finding a more intricate structure involving different behavior in different sectors. This structure is (an aspect of) the Stokes phenomenon.

1.2.4. *Stokes and anti-Stokes.*

- We call rays with $z^{\frac{3}{2}} \in \mathrm{i}\mathbb{R}$ *anti-Stokes rays*; they occur at arguments $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$; they are the loci where the constituent functions $\psi^{(i)}$ exchange dominance.
- We call rays with $z^{\frac{3}{2}} \in \mathbb{R}$ *Stokes rays*; they occur at arguments $0, \frac{2\pi}{3}, \frac{4\pi}{3}$; they are the loci at which the dominant constituent function $\psi^{(i)}$ can be discontinuous, and consequently they are also the loci at which the coefficients in the decomposition of $\mathrm{Ai}(z)$ into the $\psi^{(i)}$ can be discontinuous.

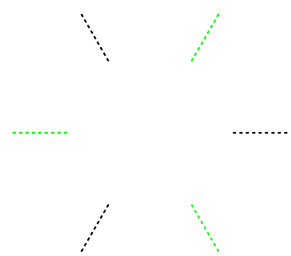

FIGURE 3. The Stokes rays (black) and anti-Stokes rays (green) for the Airy function.

(Warning: this terminology is not completely standard; some authors exchange Stokes and anti-Stokes.)

1.2.5. *Cousins of the Airy function.* We have seen that the Airy function should not be considered by itself: rather it should be considered together with various "cousins". All these functions sit naturally in a 2-dimensional vector space V. There are various different ways to understand this:

(1) We consider integrals of the form

$$\psi_\gamma(z) = \int_\gamma \mathrm{e}^{-W(s)} \mathrm{d}s, \qquad W(s) = -\mathrm{i}(\frac{1}{3}s^3 + zs) \tag{1.11}$$

where $\gamma \subset \mathbb{C}$ is any contour without boundary for which the integral is convergent (i.e. $\mathrm{Re}\, W \to +\infty$ sufficiently quickly along the contour). In this way we obtain three functions ψ_{γ_m} corresponding to the three contours γ_m shown in the figure.

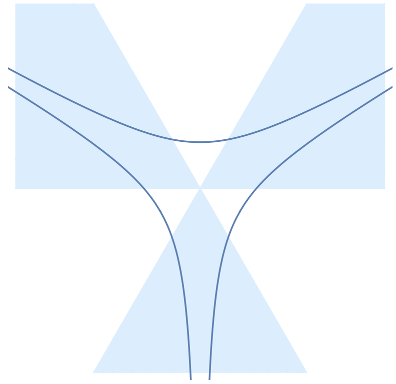

FIGURE 4. Three contours γ_m, chosen so that the integral (1.11) is convergent. Shading denotes regions where $\operatorname{Im}(s^3) > 0$; thus $\operatorname{Re} W \to +\infty$ along any contour which lies asymptotically in one of these regions, and thus integrals (1.11) along the indicated contours are absolutely convergent. Note that the integral is invariant under contour deformation as long as it remains convergent: thus it depends only on the choice of initial and final region.

These functions obey one linear relation and thus span a 2-dimensional vector space V. The $\psi^{(i)}$ are some linear combinations of the ψ_{γ_m}, with piecewise-constant coefficients.

(2) We consider the complex ODE

$$\left(\partial_z^2 - z\right)\psi(z) = 0. \tag{1.12}$$

Since this is a linear ODE of order 2 with no singularities in the plane, its global solutions in the plane form a 2-dimensional vector space V. The $\psi^{(i)}$ in each sector are the restrictions of some particular elements of V.

(3) We consider the formal series

$$\psi^{(1)}_{\text{formal}} = \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}} \sum_{n=0}^{\infty} c_n (-1)^n z^{-\frac{3}{2}n}, \tag{1.13}$$

$$\psi^{(2)}_{\text{formal}} = \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{\frac{2}{3}z^{\frac{3}{2}}} \sum_{n=0}^{\infty} c_n z^{-\frac{3}{2}n}, \tag{1.14}$$

$$c_n = \left(\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!}. \tag{1.15}$$

The series $\psi^{(i)}_{\text{formal}}$ have zero radius of convergence, but it is nevertheless possible to construct actual functions out if them, via the procedure of Borel summation. To be more precise, this procedure produces piecewise-analytic functions (honestly analytic away from the Stokes rays) with the asymptotic expansion

$$\psi^{(i)} \sim \psi^{(i)}_{\text{formal}} \qquad \text{as } z \to \infty. \tag{1.16}$$

Very loosely speaking, we may understand these cousins of the Airy function as being obtained by a kind of "generalized analytic continuation" of the function $\operatorname{Ai}(z)$: one goes very close to the singularity $z = \infty$ in one sector and then comes out again in another sector. One can also think of this as "generalized monodromy" of the equation (1.12). (Don't worry if this remark doesn't make sense yet.)

Exercise 1.3. Prove that any function $\psi_\gamma(z)$ defined by (1.11) is a solution of (1.12).

Exercise 1.4. Prove that the functions $\psi_\gamma(z)$ defined by (1.11) span the space of solutions of (1.12).

Exercise 1.5. Prove that the series (1.13) is a formal solution of (1.12).

1.3. **Topology of connections with irregular singularity.** As we just saw, the function $\mathrm{Ai}(z)$ is a solution of the complex linear ODE (1.12). Its Stokes phenomena can be understood as an aspect of the theory of complex linear ODEs — or more generally flat connections — with irregular singularity. [add refs here]

1.3.1. *Monodromy.* We consider the surface

$$C = \mathbb{CP}^1, \tag{1.17}$$

with marked points ("punctures") at $z_1, \dots, z_k$, possibly including ∞.

Now consider an order N linear scalar ODE of the form

$$\sum_{n=0}^{N} f_{N-n}(z)\partial_z^n \psi(z) = 0 \tag{1.18}$$

where $f_0(z) = 1$, and the coefficient functions $f_n(z)$ are meromorphic, with poles only at the punctures (and an additional condition governing the decay at $z = \infty$, if $z = \infty$ is not a puncture). In this situation one can extract some topological content: the solutions form a local system[5] of N-dimensional vector spaces over the topological surface

$$S = \mathbb{CP}^1 \setminus \{z_1, \dots, z_k\} \tag{1.19}$$

In particular, one obtains a representation

$$\rho : \pi_1(S) \to G \tag{1.20}$$

up to equivalence, where

$$G = \mathrm{GL}(N, \mathbb{C}). \tag{1.21}$$

We can generalize this in two ways:

- First, writing
$$\tilde{\psi}(z) = (\psi(z), \partial_z \psi(z), \dots, \partial_z^{N-1}\psi(z)) \tag{1.22}$$
the equation (1.18) is equivalent to the first-order matrix equation

$$(\partial_z + A_z(z))\tilde{\psi}(z) = 0, \qquad A_z(z) = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ f_N(z) & f_{N-1}(z) & f_{N-2}(z) & \cdots & f_1(z) \end{pmatrix}. \tag{1.23}$$

[5] See Definition B.12.

Exercise 1.6. Verify the equivalence between (1.23) and (1.18).

The equation (1.23) says $\tilde{\psi}(z)$ is a covariantly constant section of the trivial bundle $\mathcal{O}^N$ with meromorphic connection

$$\nabla = \partial + A. \tag{1.24}$$

Now the generalization is to consider a general meromorphic connection on a rank N holomorphic vector bundle over C, again with poles only at the punctures.

- For a further generalization, instead of $C = \mathbb{CP}^1$ we may consider a general Riemann surface C with punctures $z_1, \ldots, z_k$, and denote the underlying topological surface $S = C \setminus \{z_1, \ldots, z_k\}$. Then we consider general meromorphic connections on rank N holomorphic vector bundles over C, again with poles only at the punctures.

Given a meromorphic connection on a rank N holomorphic vector bundle over C, again with poles only at the punctures, we obtain a local system of N-dimensional vector spaces over S, and a monodromy representation $\pi_1(C) \to \mathrm{GL}(N, \mathbb{C})$ up to equivalence.

Now we can formulate a natural question: can we recover a meromorphic connection from its monodromy representation?

1.3.2. *Regular versus irregular.* The answer to this question will hinge on the nature of the singularities of the connection. We briefly recall the distinction between regular and irregular singularities, beginning with the case $N = 1$. The equation

$$\left(\partial_z - \frac{m}{z}\right)\psi(z) = 0 \tag{1.25}$$

(simple pole) has local solutions of the form

$$\psi(z) = cz^m. \tag{1.26}$$

Thus the solutions have power-law growth as $z \to 0$. This is the hallmark of an ODE with a regular singularity at $z = 0$. This equation also has simple monodromy on a counterclockwise loop around $z = 0$, given by the operator

$$M\psi = \mathrm{e}^{2\pi \mathrm{i} m}\psi. \tag{1.27}$$

Exercise 1.7. Verify that the monodromy of (1.25) on a counterclockwise loop around $z = 0$ is given by (1.27).

In contrast, the equation

$$\left(\partial_z - \frac{m}{z} + \frac{a}{z^2}\right)\psi(z) = 0 \tag{1.28}$$

(order 2 pole) has solutions of the form

$$\psi(z) = cz^m \exp(a/z). \tag{1.29}$$

These solutions have exponential growth and decay as $z \to 0$, controlled by the coefficient a (and again monodromy (1.27) around $z = 0$.) This is an ODE with irregular singularity at $z = 0$.[6]

[6] Although this ODE has irregular singularity, its solutions do not have Stokes phenomenon: that never occurs for first-order linear homogeneous scalar ODE.

There are similar notions of regular and irregular singularities for higher-order equations, and for connections in holomorphic bundles. These will be discussed at some length in later chapters.

1.3.3. *Riemann-Hilbert correspondence with regular singularities.* Suppose we define a morphism between two meromorphic connections to be a meromorphic bundle map respecting the connections. Then, we have two categories

$$\mathcal{C}_{\mathrm{dR}}(G,C) = \text{meromorphic } \mathrm{GL}(N,\mathbb{C})\text{-connections over } C \text{ with regular singularities,} \tag{1.30}$$

$$\mathcal{C}_{\mathrm{Betti}}(G,S) = \text{rank } N \text{ local systems over } S, \tag{1.31}$$

and taking covariantly constant sections gives a functor

$$\mathrm{Sol} : \mathcal{C}_{\mathrm{dR}}(G,C) \to \mathcal{C}_{\mathrm{Betti}}(G,S)\,. \tag{1.32}$$

This functor is actually an equivalence of categories, and induces an isomorphism between the moduli spaces of objects,[7] the monodromy map

$$\mathrm{Mon} : \mathcal{M}_{\mathrm{dR}}(G,C) \to \mathcal{M}_{\mathrm{Betti}}(G,S)\,. \tag{1.33}$$

Moreover $\mathcal{M}_{\mathrm{Betti}}(G,S)$ is a nice concrete space:

$$\mathcal{M}_{\mathrm{Betti}}(G,S) = \{\text{representations } \rho : \pi_1(S) \to G\}/\sim\,. \tag{1.34}$$

From here you can see at once that it is a complex space, or an algebraic variety. This is already interesting, but beginning from (1.33) one can discover much more structure on this space. In particular, (after restricting a bit by imposing a "stability condition" and fixing the eigenvalues of monodromy around punctures),

- it has a natural holomorphic symplectic structure,
- one can quantize it (skein algebras),
- it is diffeomorphic to a moduli space of Higgs bundles,
- it has a nice differential geometry (hyperkähler metric),
- there are deeper things (SYZ mirror symmetry, $P = W$ conjecture...)

1.3.4. *Riemann-Hilbert correspondence with irregular singularities.* Now how about connections with irregular singularities? If we just replace "regular" with "irregular" in the definition of $\mathcal{C}_{\mathrm{dR}}(G,C)$, we will still have a well defined Sol functor, but it is not an equivalence anymore: it forgets some structure of the connection, associated with the behavior around the irregular singularities. Said more sharply: it can happen that there are two connections which have the same monodromy but are not related by a meromorphic bundle map.

However, there is an upgraded version which is an equivalence. It involves new categories on both sides: a category $\mathcal{C}_{\mathrm{dR}}(G,C,\Theta)$ of connections with irregular singularity, where Θ is some additional data at the punctures (like extra moduli), and a category $\mathcal{C}_{\mathrm{Betti}}(G,\widehat{S},\theta)$ of "Stokes local systems," where $(\widehat{S},\theta)$ is S with some additional topological data at the punctures. Then we have as before

$$\mathrm{Sol} : \mathcal{C}_{\mathrm{dR}}(G,C,\Theta) \to \mathcal{C}_{\mathrm{Betti}}(G,\widehat{S},\theta)\,, \tag{1.35}$$

[7] The right language might be that it is a map of complex analytic stacks.

and a corresponding map of moduli spaces (irregular Riemann-Hilbert map)

$$\mathrm{Sol} : \mathcal{M}_{\mathrm{dR}}(G, C, \Theta) \to \mathcal{M}_{\mathrm{Betti}}(G, \widehat{S}, \theta)\,. \tag{1.36}$$

As in the case of regular singularities, one finds that the spaces $\mathcal{M}_{\mathrm{Betti}}(G, \widehat{S}, \theta)$ get equipped in this way with a lot of extra structure. This is particularly nice because these spaces are in many cases *simpler* than their cousins with regular or no singularities: in particular they allow us to isolate phenomena that have to do more with G and less with $\widehat{S}$.

1.3.5. *Schrödinger equations with polynomial potential.* A fundamental example, to which we will return repeatedly, arises when we take

$$C = \mathbb{CP}^1 \tag{1.37}$$

with a single puncture at $z = \infty$. In this case $\pi_1(S)$ is trivial — said otherwise, there is no possibility of monodromy, so a local system over S contains no information (except for the rank).

Nevertheless, there is some interesting topological structure to be uncovered, if we study connections on C with an irregular singularity at $z = \infty$. For example, suppose we consider Schrödinger equations with polynomial potential:

$$\left(\partial_z^2 + P(z)\right) \psi(z) = 0\,, \tag{1.38}$$

where $P(z)$ is a polynomial of degree n.[8] Then we have a picture similar to the one we discussed for the Airy equation (the case $P(z) = -z$), as follows.

The equation (1.38) has a 2-dimensional space V of global solutions. There are $n + 2$ Stokes rays going into the irregular singularity at $z = \infty$. Thus we let $\widehat{S}$ be the real blow-up of $\mathbb{CP}^1$ at $z = \infty$, which is homeomorphic to a disc. This disc naturally carries $n + 2$ marked points on the boundary, corresponding to the Stokes rays.[9]

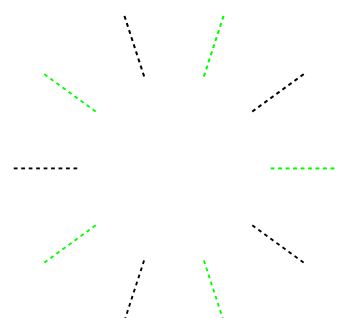

FIGURE 5. The Stokes rays (black) and anti-Stokes rays (green) for a Schrödinger equation with cubic potential $P(z) = z^3 + az^2 + bz + c$.

For each Stokes ray ℓ_i, there is a 1-dimensional subspace $L_i \subset V$, consisting of solutions which decay exponentially along ℓ_i. This is the data of a Stokes local system over $(\widehat{S}, \theta)$: it amounts to a 2-dimensional vector space V, *plus* the distinguished subspaces $L_i \subset V$,

[8] By the change of variables $w = 1/z$ you can check that there is indeed an irregular singularity at $z = \infty$. The condition for the singularity to be regular would be that $P(z)$ decays quadratically as $z \to \infty$. This can't happen for a polynomial, but it can happen for a rational function.

[9] Thus the mapping class group of $\widehat{S}$ is $\mathbb{Z}/(n + 2)\mathbb{Z}$, acting by rotations.

with the condition $L_{i+1} \neq L_i$. The moduli space of these Stokes local systems maps to the configuration space of $(n+2)$-tuples of points in $\mathbb{CP}^1$,

$$\mathcal{M}_{\text{Betti}}(G, \widehat{S}, \theta) \hookrightarrow \text{Conf}_{n+2}(\mathbb{CP}^1) = (\mathbb{CP}^1)^{n+2}/\text{PGL}(2, \mathbb{C}) \tag{1.39}$$

and this map is surjective onto a dense subset. (In the case $n = 1$, there is a unique Stokes local system, i.e. $\mathcal{M}_{\text{Betti}}(G, \widehat{S}, \theta)$ is a point. So the Airy equation is not sitting in an interesting moduli space. As we will see, the first really interesting example is $n = 3$, where we get the 2-dimensional space $\text{Conf}_5^{\text{cyc}}(\mathbb{CP}^1)$.)

The decorated surface (C, Θ) involves a bit more information. Let's take n odd for simplicity. Then the data determining Θ is a tuple of complex numbers $(c_{\frac{1}{2}(n-1)}, \ldots, c_n)$. For each such tuple one defines a moduli space $\mathcal{M}_{\text{dR}}(G, C, \Theta)$ of meromorphic connections on $\mathbb{CP}^1$ with irregular singularity at $z = \infty$. It contains as an open dense subset the set of connections on $\mathcal{O}^2$ of the form

$$\nabla_z = \partial_z + A_z(z) \tag{1.40}$$

with $A_z(z) \in \mathfrak{gl}(2)$ such that $\text{Tr}\, A_z(z) = 0$, and $\det A_z(z) - \sum_{k=\frac{1}{2}(n-1)}^{n} c_k z^k$ is a polynomial of degree $< \frac{1}{2}(n-1)$. This includes Schrödinger equations (1.38) as a subspace of dimension $\frac{1}{2}(n-1)$, but also contains other connections. [check this is correct description and the space has total dimension $n-1$]

1.3.6. *Dual Poisson-Lie group.* Another notable example arises if we take C to be $\mathbb{CP}^1$ and consider equations with one regular singularity (say $z = 0$) and one irregular singularity (say $z = \infty$), with appropriate asymptotic conditions. Concretely, say we fix $u \in \mathfrak{gl}(N, \mathbb{C})$ and consider connections on $\mathcal{O}^N$ of the form

$$\nabla_z = \partial_z + u + \frac{a}{z}, \tag{1.41}$$

with $a \in \mathfrak{gl}(N, \mathbb{C})$. These connections make up an open dense subset in a moduli space $\mathcal{M}_{\text{dR}}(G, C, \Theta)$ of connections with irregular singularity. Now (roughly speaking — the precise statement is in [2]) the corresponding moduli space $\mathcal{M}_{\text{Betti}}(G, \widehat{S}, \theta)$ of Stokes data turns out to be the *dual Poisson-Lie group* G^*, and the Riemann-Hilbert map (1.36) induces the *linearization* map, relating the Poisson structure of G^* to the usual Kirillov-Kostant structure on $\mathfrak{g}^*$.[10]

1.4. **The WKB method.** Here is a second important place where Stokes phenomenon occurs. Suppose we want to actually *compute* the Riemann-Hilbert map in practice, in any of the examples we have just discussed (either with no singularities, regular singularities or irregular singularities). This requires us to get some practical handle on the flat sections of a connection.

1.4.1. *The formal WKB series.* In the case $N = 1$, computing the flat sections boils down to integration: the solutions of

$$(\partial_z - \lambda(z))\psi(z) = 0 \tag{1.42}$$

are of the form

$$\psi(z) = \exp \int_{z_0}^{z} \lambda(z')\, \mathrm{d}z'. \tag{1.43}$$

[10] I learned about this from Sasha Goncharov, Xiaomeng Xu and Yan Zhou.

Our main interest is in the $N > 1$ case, e.g. the equation (1.38),

$$\left(\partial_z^2 + P(z)\right) \psi(z) = 0\,. \tag{1.44}$$

In this case one cannot directly write down an integral formula for a solution $\psi(z)$.

Remarkably, though, one can get a handle on the solutions by another instance of Stokes's trick of "expanding around an irregular singularity." The irregular singularity in question appears only after we introduce a new parameter $\hbar \in \mathbb{C}^\times$: we replace (1.44) by

$$\left(\partial_z^2 + \hbar^{-2} P(z)\right) \psi(z, \hbar) = 0 \tag{1.45}$$

and consider the expansion of the solutions $\psi(z, \hbar)$ around $\hbar = 0$. Then one can write formal solutions in the form

$$\psi^{(i)}_{\mathrm{formal}}(z, \hbar) = \exp\left(\hbar^{-1} \int_{z_0}^{z} \lambda^{(i)}_{\mathrm{formal}}(z', \hbar) \mathrm{d}z'\right) \tag{1.46}$$

where $\lambda^{(i)}_{\mathrm{formal}}(z, \hbar) = \sqrt{-P(z)} + \cdots$ is a formal series solution of the Riccati equation

$$\lambda^2 + P + \hbar \partial_z \lambda = 0. \tag{1.47}$$

and the index i labels the choice of $\sqrt{-P(z)}$. The first few orders are

$$\lambda^{(i)}_{\mathrm{formal}} = \sqrt{-P} - \hbar \frac{P'}{4P} + \hbar^2 \sqrt{-P} \frac{5P'^2 - 4PP''}{32P^3} + \cdots \tag{1.48}$$

Exercise 1.8. Verify that the Riccati equation (1.47) is the condition for (1.46) to solve (1.45).

However, the series $\lambda^{(i)}_{\mathrm{formal}}$ is never convergent (except when P is constant).

1.4.2. *The exact WKB method.* One then faces the problem of promoting these formal solutions to actual solutions. This can indeed be done (at least in sufficiently nice cases, including Schrödinger equations with polynomial potential), using the device of Borel summation. The resulting exact local solutions $\psi^{(i)}$ are analogues of the local solutions $\psi^{(i)}$ which we discussed for the Airy equation; they are piecewise smooth, with uniform asymptotic series expansion as $\hbar \to 0$. They jump at the Stokes rays, and this leads to some interesting geometry.

For instance, one can fix a phase ϑ and ask: is the ray $\arg \hbar = \vartheta$ a Stokes ray? The answer depends on the value of z (recall that we are considering a formal series in $\hbar$, whose coefficients depend on z.) The set of z for which the answer is positive is called the *Stokes graph* or *spectral network* $\mathcal{W}(\vartheta)$. For an example see Figure 6.

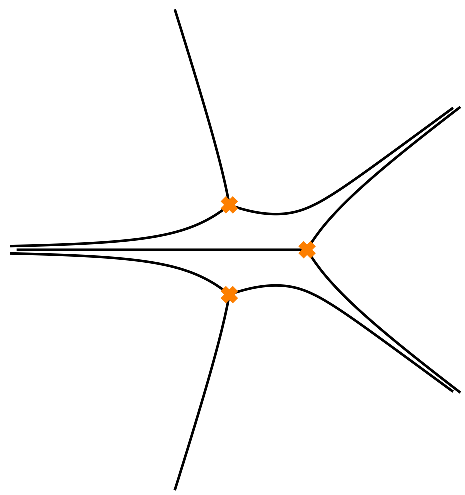

FIGURE 6. The Stokes graph $\mathcal{W}(\vartheta = 0)$ for the polynomial Schrödinger equation (1.45) with potential $P(z) = z^3 - 1$. The orange crosses mark the roots of $P(z)$.

If we are considering connections which have irregular singularities, then there is a pleasant coherence between the two sorts of Stokes phenomena we have discussed: for example, note that the Stokes graph shown in Figure 6 (governing the $\hbar \to 0$ behavior at fixed z) is asymptotic to the Stokes rays shown in Figure 5 (governing the $z \to \infty$ behavior at fixed $\hbar$).

[comment about relations to clusters, ...]

2. CONNECTIONS WITH REGULAR SINGULARITIES

Let C be a Riemann surface (not necessarily compact), with a finite subset $D \subset C$. As before let $G = \mathrm{GL}(N, \mathbb{C})$.

2.1. Meromorphic connections and gauge transformations.

We want to study meromorphic connections of rank N over C, with poles only at D. The basic examplar of such an object is an operator of the form

$$\nabla = \mathrm{d} + A, \tag{2.1}$$

or in longhand

$$\nabla\psi = (\partial_z + A_z(z))\psi\, \mathrm{d}z \tag{2.2}$$

where $\psi : U \to \mathbb{C}^N$ and $A_z : U \to \mathfrak{gl}(N, \mathbb{C})$ are meromorphic, with poles only at D.

If we let:

- $\mathcal{O}_C(*D)(U) =$ the space of meromorphic functions on U with poles only at D,
- $\Omega^1_C(*D)(U) =$ the space of meromorphic 1-forms on U with poles only at D,
- $\mathcal{F}(U) = \mathcal{O}_C(*D)(U)^N$,

then ∇ defined by (2.2) is a map

$$\nabla : \mathcal{F}(U) \to \mathcal{F}(U) \otimes_{\mathcal{O}_C(*D)(U)} \Omega^1_C(*D)(U), \tag{2.3}$$

obeying the Leibniz rule

$$\nabla(f\psi) = \psi \otimes \mathrm{d}f + f\nabla\psi, \qquad f \in \mathcal{O}_C(*D)(U), \quad \psi \in \mathcal{F}(U). \tag{2.4}$$

Exercise 2.1. Prove that every such ∇ is of the form (2.2). (Hint: by the Leibniz rule ∇ is determined by its action on the sections $(0, 0, \ldots, 1, \ldots, 0)$.)

These maps are compatible with restrictions from U to a smaller open set; a fancy way of saying this is that they can be organized into a map of sheaves of $\mathcal{O}_C(*D)$-modules (see §B.1 for the definitions),

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_C(*D)} \Omega^1_C(*D). \tag{2.5}$$

Once said this way, there is a natural generalization: we could replace $\mathcal{F}$ by any locally free sheaf of $\mathcal{O}_C(*D)$-modules (i.e. $\mathcal{F}$ should be isomorphic to $\mathcal{O}_C(*D)^N$ locally but maybe not globally.) Such an $\mathcal{F}$ is also called a meromorphic bundle with poles at D. Thus we arrive at our notion of a meromorphic connection:[11]

Definition 2.1 (Meromorphic connection). A *meromorphic connection* over C with poles only at D is a pair $(\mathcal{F}, \nabla)$, where $\mathcal{F}$ is a locally free sheaf of $\mathcal{O}_C(*D)$-modules, and ∇ is a map of sheaves

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_C(*D)} \Omega^1_C(*D) \tag{2.6}$$

obeying the Leibniz rule

$$\nabla(fs) = s \otimes \mathrm{d}f + f\nabla s. \tag{2.7}$$

Definition 2.2 (Morphism of meromorphic connections). A morphism of meromorphic connections $(\mathcal{F}, \nabla) \to (\mathcal{F}', \nabla')$ is an isomorphism $g : \mathcal{F} \to \mathcal{F}'$ of sheaves of $\mathcal{O}_C(*D)$-modules, which carries ∇ into ∇', i.e. $g(\nabla\psi) = \nabla'(g\psi)$.

Example 2.3 (Morphisms of connections on the trivial bundle). If $\mathcal{F} = \mathcal{F}' = \mathcal{O}_C(*D)^N$, then a morphism $g : \mathcal{F} \to \mathcal{F}'$ means a meromorphic map $g : C \to \mathrm{Mat}_{N\times N}(\mathbb{C})$ whose determinant is not identically zero. The condition that g carries $\nabla = \mathrm{d} + A$ into $\nabla' = \mathrm{d} + A'$ is that

$$A'_z = gA_z g^{-1} + g\partial_z g^{-1}. \tag{2.8}$$

Exercise 2.2. Verify (2.8).

Such a g is sometimes called a "gauge transformation," and (2.8) is called the "gauge transformation law for connections." If $N = 1$, or more generally if g commutes with A_z and $\partial_z g$, then (2.8) becomes particularly simple:

$$A'_z = A_z + g\partial_z g^{-1} = A_z - \partial_z \log g. \tag{2.9}$$

2.2. **Regular singularities.** Now we introduce the class of connections with regular singularities.

[11] In higher dimensions we would want to impose a flatness condition, but in dimension 1 the flatness is automatic.

Definition 2.4 (Meromorphic connection with regular singularities). We say a meromorphic connection $(\mathcal{F}, \nabla)$ has *regular singularities* if, for each $p \in D$, there is some neighborhood of p in which $(\mathcal{F}, \nabla)$ is isomorphic to a connection $(\mathcal{O}_C^N(*D), \nabla = \mathrm{d} + A)$ where A_z has at worst a first-order pole at p.

Example 2.5 (Basic example of regular singularity). The prototypical example of a connection with regular singularities is

$$\nabla_z = \partial_z - \frac{m}{z} \tag{2.10}$$

with $m \in \mathfrak{gl}(N, \mathbb{C})$. (Here and below, when we do not mention C or $\mathcal{F}$ explicitly we assume C is a disc containing 0 and $\mathcal{F} = \mathcal{O}_C^N(*0)$.)

Exercise 2.3. Show that the monodromy of the connection (2.10) around a counterclockwise loop around $z = 0$ is $M = \exp(2\pi \mathrm{i} m) \in G$.

Exercise 2.4. Consider a connection with regular singularities of the form

$$\nabla_z = \partial_z - \frac{m}{z} + B(z) \tag{2.11}$$

where $B(z)$ is holomorphic. Let M be the monodromy around a counterclockwise loop around $z = 0$, relative to some basepoint z_0. If $\mathcal{O}_M^\times \subset G$ is the adjoint orbit containing M, show that the closure $\overline{\mathcal{O}_M^\times}$ contains $\exp(2\pi \mathrm{i} m)$. In particular, the generalized eigenvalues of M and m are related by $\lambda_M^{(j)} = \exp(2\pi \mathrm{i} \lambda_m^{(j)})$, and if no two $\lambda_m^{(j)}$ differ by an integer, then M is conjugate to $\exp(2\pi \mathrm{i} m)$. [check]

Example 2.6 (Second-order pole but regular singularity). A subtler example is

$$\nabla_z = \partial_z + \frac{B}{z^2} \tag{2.12}$$

where $B \in \mathfrak{gl}(N, \mathbb{C})$ is any nilpotent matrix. Indeed, in this case $g = \mathrm{e}^{-B/z}$ is actually meromorphic, and a gauge transformation by this g relates ∇ to $\nabla'_z = \partial_z$, so ∇ has regular singularities despite the second-order pole.

This example is meant to illustrate that it is not necessarily easy to recognize whether a connection has regular singularities or not.

Here is another way to codify the definition of regular singularities. Let $\Omega_C^1(D)$ be the sheaf of 1-forms with at most first-order poles at D. Then:

Proposition 2.7 (Alternative description of meromorphic connections with regular singularities). A meromorphic connection $(\mathcal{F}, \nabla)$ has regular singularities just if there exists a locally free sheaf $\overline{\mathcal{F}} \subset \mathcal{F}$ of $\mathcal{O}_C$-modules, with $\overline{\mathcal{F}} = \mathcal{F}$ on $C \setminus D$, and $\nabla(\overline{\mathcal{F}}) \subset \overline{\mathcal{F}} \otimes_{\mathcal{O}_C} \Omega_C^1(D)$.

Exercise 2.5. Prove Proposition 2.7.

This second version is closer to what one uses in higher dimensions to study regular singularities. (See [3] for a very readable and short account.) But here we will stick to dimension 1.

Exercise 2.6. (From [4]. [might be hard]) Consider a meromorphic connection $(\mathcal{O}_C^N(*0), \nabla)$ with Laurent expansion

$$\nabla_z = \partial_z + \frac{B_{-r-1}}{z^{r+1}} + \cdots + \frac{B_{-1}}{z} + B_0 + \cdots \tag{2.13}$$

where $r \geq 1$ and $B_{-r-1} \neq 0$. If ∇ has a regular singularity at $z = 0$, prove that B_{-r-1} is nilpotent, and $\operatorname{Tr} B_{-r-1} = \cdots = \operatorname{Tr} B_{-2} = 0$.

To finish this section we summarize the structure we have defined in the form of a category:

Definition 2.8 (de Rham category with regular singularities). Let $\mathcal{C}_{\mathrm{dR}}(G, C, D)$ be the category of meromorphic connections over C, of rank N, with regular singularities in D.

If $D = \emptyset$ then we also write this category as $\mathcal{C}_{\mathrm{dR}}(G, C)$.

2.3. **Local systems of solutions.** Now suppose given a meromorphic connection $(\mathcal{F}, \nabla)$ of rank N over C, with poles only along D. Let $S = C \setminus D$. For any open $U \subset S$ define

$$\operatorname{Sol}(\mathcal{F}, \nabla)(U) = \{\psi \in \mathcal{F}(U) : \nabla\psi = 0\}. \tag{2.14}$$

Proposition 2.9 (Solutions on simply connected domains). If U is open and simply connected, then $\operatorname{Sol}(\mathcal{F}, \nabla)(U)$ is an N-dimensional vector space. If in addition $V \subset U$ is open and connected, then the restriction map $r_{UV} : \operatorname{Sol}(\mathcal{F}, \nabla)(U) \to \operatorname{Sol}(\mathcal{F}, \nabla)(V)$ is an isomorphism.

Proof. We can cover U by simply connected open subsets on which $\mathcal{F} \simeq \mathcal{O}^N$, with simply connected intersections. This reduces to the case where $\mathcal{F} \simeq \mathcal{O}^N$. (In fact, $\mathcal{F} \simeq \mathcal{O}^N$ on the whole U anyway, so this reduction is actually unnecessary — but we do not need to use this.) Then we may as well assume $\mathcal{F} = \mathcal{O}^N$ and $U \subset \mathbb{C}$, so $\operatorname{Sol}(\mathcal{F}, \nabla)(U)$ is the set of ψ obeying

$$(\partial_z + A_z(z))\psi(z) = 0, \tag{2.15}$$

for some holomorphic $A_z : U \to \mathfrak{gl}(N, \mathbb{C})$. Choose some $z_0 \in U$. It follows from the existence and uniqueness theorem for complex linear ODE that, for any $\psi_0 \in \mathbb{C}^N$, there exists a unique solution $\psi(z)$ of (2.15) with $\psi(z_0) = \psi_0$. This gives the desired isomorphism

$$\operatorname{Sol}(\mathcal{F}, \nabla)(U) \simeq \mathbb{C}^N. \tag{2.16}$$

If we also assume that $z_0 \in V$ then the statement about restrictions also follows directly. □

We remark that the isomorphism obtained in the proof above is not a canonical isomorphism, since it depends on our choice of an isomorphism $\mathcal{F} \simeq \mathcal{O}^N$ and basepoint z_0. What is canonical is that, for any $p \in U$,

$$\operatorname{Sol}(\mathcal{F}, \nabla)(U) \simeq \operatorname{Sol}(\mathcal{F}, \nabla)_p. \tag{2.17}$$

We can recast Proposition 2.9 in more categorical language as follows. We recall the notion of a local system, Definition B.12. In short, the constant sheaf $\underline{Z}$ is

$$\underline{Z}(U) = \{\text{locally constant maps } U \to Z\}, \tag{2.18}$$

and a local system is a sheaf which is locally isomorphic to a constant sheaf.

Proposition 2.10 (Solutions form a local system). $\operatorname{Sol}(\mathcal{F}, \nabla)$ is a local system over S valued in N-dimensional $\mathbb{C}$-vector spaces.

Proof. For any $p \in S$ choose a simply connected open $U \subset S$ containing p. Let

$$Z = \mathrm{Sol}(\mathcal{F}, \nabla)(U). \tag{2.19}$$

By Proposition 2.9, Z is an N-dimensional vector space. For any open $V \subset U$, applying Proposition 2.9 to each connected component of V, we see that $\mathrm{Sol}(\mathcal{F}, \nabla)(V)$ is the space of locally constant maps $V \to Z$. This means precisely that $\mathrm{Sol}(\mathcal{F}, \nabla)|_U \simeq \underline{Z}$. □

Definition 2.11 (Betti category with regular singularities). Let $\mathcal{C}_{\mathrm{Betti}}(G, S)$ be the category of local systems over S valued in rank N $\mathbb{C}$-vector spaces (defined in Definition B.12), where for morphisms we take only isomorphisms of local systems. (Thus this category is a groupoid: all morphisms are invertible.)

2.4. **Moderate growth.** Here is a key property of connections with regular singularities (which actually is sometimes used as the definition).

Definition 2.12 (Moderate growth). Let $U \subset C$ be any open set with $p \in \overline{U}$, and let z be any local coordinate on a neighborhood of $\overline{U}$ centered at p.

- A function f on U has *moderate growth* at p if $|f| \ll |z|^K$ as $z \to 0$ for some $K \in \mathbb{R}$.
- Suppose $\mathcal{F}$ is a locally free sheaf of $\mathcal{O}_C(*D)$-modules. A section $\psi \in \mathcal{F}(U)$ has *moderate growth* at p if, choosing some trivialization $\iota : \mathcal{F} \to \mathcal{O}_C(*D)^N$, each component of $\iota(\psi)$ has moderate growth at p.

Proposition 2.13 (Moderate growth around regular singularities). If $(\mathcal{F}, \nabla)$ has regular singularities, then for any open set U, all $\psi \in \mathcal{F}(U)$ with $\nabla\psi = 0$ have moderate growth at any $p \in \overline{U}$.

Proof. Use a coordinate system in which p is $z = 0$, and a local trivialization for which $A_z(z)$ has only a first-order pole at $z = 0$, with residue m. Then pass to the coordinate $t = -\log z$; the equation $\nabla\psi = 0$ becomes

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = B(t)\psi \tag{2.20}$$

for a matrix $B(t)$ such that $B(t) \to m$ as $t \to +\infty$. It follows that the sup norm $\|B(t)\|$ is bounded by a constant K' in some right half-plane strip, and thus using Grönwall's lemma one sees that $\|\psi\|$ can only grow like $\mathrm{e}^{K' \operatorname{Re} t}$, i.e. like $|z|^{K'}$. Then take any $K > K'$. □

Our main use for the moderate growth property is the following:

Lemma 2.14 (Gauge transformations between connections with regular singularities are automatically meromorphic). Suppose we have meromorphic connections $(\mathcal{F}, \nabla)$ and $(\mathcal{F}', \nabla')$ over C with regular singularities at D, and a morphism

$$g : (\mathcal{F}, \nabla)|_{C \setminus D} \to (\mathcal{F}', \nabla')|_{C \setminus D}. \tag{2.21}$$

Then g extends over C.

Proof. The problem is local, so we may as well assume $\mathcal{F}$, $\mathcal{F}'$ are trivial. Over any simply connected $U \subset C \setminus D$, we may choose an isomorphism $\underline{\mathbb{C}^N} \to \mathrm{Sol}(\mathcal{O}^N, \nabla)$ and tensor with $\mathcal{O}_C$ to get

$$\Psi : \mathcal{O}^N \to \mathcal{F}\,. \tag{2.22}$$

(A Ψ obtained in this way is sometimes called a "fundamental solution" for $(\mathcal{O}^N, \nabla)$ since the vectors $(\Psi(1,0,0,\dots,0), \Psi(0,1,0,\dots,0), \dots, \Psi(0,0,0,\dots,1))$ give a basis for $\mathrm{Sol}(\mathcal{O}^N, \nabla)$.) Then

$$\Psi' = g \circ \Psi \tag{2.23}$$

is similarly a fundamental solution of ∇'. Rearrange this to

$$g = \Psi' \circ \Psi^{-1}. \tag{2.24}$$

Now, the entries of Ψ' and Ψ all have moderate growth at any point of D (using Proposition 2.13 and the fact that the columns are solutions). The same is true for Ψ^{-1} since its entries are rational functions of the entries of Ψ. Thus the entries of $g|_U$ also have moderate growth at any point of D.

Since U was an arbitrary simply connected subset, we conclude that g has moderate growth at any point of D. Then the Riemann removable singularities theorem says that g extends meromorphically to C. □

2.5. **Riemann-Hilbert map.** Now we can formulate our main theorem about connections with regular singularities. We consider the functor

$$\mathrm{Sol} : \mathcal{C}_{\mathrm{dR}}(G, C, D) \to \mathcal{C}_{\mathrm{Betti}}(G, S) \tag{2.25}$$

which takes a meromorphic connection $(\mathcal{F}, \nabla)$ with regular singularities to its local system of solutions $\mathrm{Sol}(\mathcal{F}, \nabla)$.

Exercise 2.7. Complete the construction of the functor Sol. (We have given a map on objects; the main point is to give the maps on morphisms, i.e. given a morphism of meromorphic connections you need to construct a corresponding morphism of local systems.)

Exercise 2.8. Show that the functor Sol is full and faithful. (This amounts to showing that for any pair of connections, the map from morphisms of connections to morphisms of local systems is a bijection. For the fullness you will need Lemma 2.14.)

What we want to say is roughly that the functor Sol is invertible: a meromorphic connection with regular singularities is "the same data" as a local system. One precise formulation of this is:

Theorem 2.15 (Riemann-Hilbert map is an equivalence). The functor Sol is an equivalence of categories.

Proof. It is sufficient to show that Sol is full, faithful and essentially surjective. The first two parts are Exercise 2.8. What remains is the essential surjectivity: given a local system L of N-dimensional vector spaces on S, we need to show that L is isomorphic to some $\mathrm{Sol}(\mathcal{F}, \nabla)$.

Away from the singularities, the construction is tautological: over $C \setminus D$ we consider the $\mathcal{O}_C$-module

$$\mathcal{F}^\circ = L \otimes_{\mathbb{C}} \mathcal{O}_C \tag{2.26}$$

with the connection

$$\nabla^\circ(\psi \otimes f) = \psi \otimes \mathrm{d}f\,. \tag{2.27}$$

Note that this connection has $\mathrm{Sol}(\mathcal{F}^\circ, \nabla^\circ) \simeq L$ as needed.

What remains is to give an extension $\overline{\mathcal{F}}$ of $\mathcal{F}^\circ$ to an $\mathcal{O}_C$-module over the whole C, such that ∇° extends to a map $\overline{\mathcal{F}} \to \overline{\mathcal{F}} \otimes_{\mathcal{O}_C} \Omega^1_C(D)$. (Once this is done we will take $\mathcal{F} = \overline{\mathcal{F}} \otimes_{\mathcal{O}_C} \mathcal{O}_C(*D)$.)

For this it is sufficient to work locally around one point $p \in D$. Let U be a punctured neighborhood of p, and let $M \in \mathrm{GL}(N, \mathbb{C})$ denote the counterclockwise monodromy of L around p, relative to some basepoint z_0 and basis of L_{z_0}. Then choose a logarithm $m \in \mathfrak{gl}(N)$ obeying

$$\exp(2\pi i m) = M. \tag{2.28}$$

(Crucially, $\exp : \mathfrak{gl}(N, \mathbb{C}) \to \mathrm{GL}(N, \mathbb{C})$ is surjective, so such an m exists!) Now choose a local coordinate z on U, and consider a meromorphic connection ∇^m on $\mathcal{O}^N$ over U given by

$$\nabla^m_z = \partial_z - \frac{m}{z}\,. \tag{2.29}$$

The local system $\mathrm{Sol}(\mathcal{O}^N, \nabla^m)|_U$ has counterclockwise monodromy M, according to Exercise 2.3. Thus the local systems $\mathrm{Sol}(\mathcal{F}^\circ, \nabla^\circ)|_U$ and $\mathrm{Sol}(\mathcal{O}^N, \nabla^m)|_U$ are isomorphic (Construction B.13). It follows (using the fact that Sol is full) that over U there is an isomorphism $(\mathcal{F}^\circ, \nabla^\circ) \to (\mathcal{O}^N, \nabla^m)$. We use this isomorphism to glue the two connections (Construction 2.16). The glued connection gives the desired extension $\overline{\mathcal{F}}$ of $\mathcal{F}^\circ$ over C. □

Construction 2.16 (Gluing connections). Suppose $C = U \cup U'$ and we have meromorphic connections $(\mathcal{F}, \nabla)$, $(\mathcal{F}', \nabla')$ over U, U' respectively. Suppose we also have an isomorphism $g : (\mathcal{F}, \nabla) \to (\mathcal{F}', \nabla')$ over $U \cap U'$. Then there exists a meromorphic connection $(\hat{\mathcal{F}}, \hat{\nabla})$ over C, whose restriction to U, U' is isomorphic to $(\mathcal{F}, \nabla)$, $(\mathcal{F}', \nabla')$ respectively.

Proof. The glued sheaf $\hat{\mathcal{F}}$ is defined by

$$\hat{\mathcal{F}}(V) = \{(\psi, \psi') \in \mathcal{F}(V \cap U) \times \mathcal{F}'(V \cap U') : \psi'|_{V \cap U \cap U'} = g\psi|_{V \cap U \cap U'}\} \tag{2.30}$$

and $\hat{\nabla}$ is defined similarly. □

Combining Theorem 2.15 with Exercise B.3 we obtain:

Corollary 2.17 (Riemann-Hilbert for regular singularities induces a bijection). The map $(\mathcal{F}, \nabla) \mapsto \mathrm{Sol}(\mathcal{F}, \nabla)$ induces a bijection

$$\mathrm{Mon} : \mathcal{M}_{\mathrm{dR}}(G, C, D) \to \mathcal{M}_{\mathrm{Betti}}(G, S). \tag{2.31}$$

Exercise 2.9. Prove that there exists a unique holomorphic connection on the trivial bundle over $\mathbb{CP}^1$, and there exists no holomorphic connection on any nontrivial holomorphic bundle over $\mathbb{CP}^1$. (Hint: use Theorem 2.15.)

What about the automorphisms? A generic representation of $\pi_1(S)$ is irreducible, and as a consequence its automorphism group is $\mathrm{GL}(1, \mathbb{C}) \simeq \mathbb{C}^\times$; using our equivalence, this proves immediately that the group of meromorphic automorphisms of the corresponding connection $(\mathcal{F}, \nabla)$ is also $\mathbb{C}^\times$. For non-generic representations the automorphism groups again match up.

The moduli sets $\mathcal{M}_{\mathrm{dR}}$, $\mathcal{M}_{\mathrm{Betti}}$ are not *just* sets: rather, they are equipped with extra geometric structure (e.g. complex structure, Poisson structure, algebraic structure). We will not give a rigorous general development of this structure, but we will discuss it in examples where we can see it directly.

2.6. Some special cases.

2.6.1. *Rank* 1 *connections.* In the case of rank 1, the Riemann-Hilbert map becomes much more explicit.

In particular, let's consider connections ∇ with regular singularity, where the extension $\overline{\mathcal{F}}$ can be taken to be the trivial bundle, $\overline{\mathcal{F}} = \mathcal{O}_C$. These connections give a subspace

$$\mathcal{M}^\circ_{\mathrm{dR}}(\mathrm{GL}(1), C, D) \subset \mathcal{M}_{\mathrm{dR}}(\mathrm{GL}(1), C, D)\,. \tag{2.32}$$

Such a connection can be written in a particularly explicit way: it has the form $\nabla = \partial + A$ where $A = A_z\,\mathrm{d}z$ is a global meromorphic 1-form, with at most first-order poles at D. Thus we can write

$$\mathcal{M}^\circ_{\mathrm{dR}}(\mathrm{GL}(1), C, D) = \Omega^1_C(D)/\sim \tag{2.33}$$

where the equivalence is $A \sim A - \mathrm{d}\log g$, with $g : C \to \mathbb{C}^\times$ meromorphic, with poles only at D.[12]

To describe the corresponding local systems explicitly, we pass to the corresponding representations: that just means maps $\rho : \pi_1(S) \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^\times$, but because $\mathbb{C}^\times$ is abelian, those maps factor through maps $\rho : H_1(S, \mathbb{Z}) \to \mathbb{C}^\times$. Thus

$$\mathcal{M}_{\mathrm{Betti}}(\mathrm{GL}(1), S) = \mathrm{Hom}(H_1(S, \mathbb{Z}), \mathbb{C}^\times) = H^1(S, \mathbb{C}^\times) \tag{2.34}$$

Then we can describe the monodromy map

$$\mathrm{Mon} : \mathcal{M}^\circ_{\mathrm{dR}}(\mathrm{GL}(1), C, D) \hookrightarrow \mathcal{M}_{\mathrm{Betti}}(\mathrm{GL}(1), S) \tag{2.35}$$

in explicit terms, as follows.

Exercise 2.10 (Monodromy of rank 1 connections in the trivial bundle). Show that, for a connection $\nabla = \partial + A$, the representation

$$\rho = \mathrm{Mon}(A) : H_1(S, \mathbb{Z}) \to \mathbb{C}^\times \tag{2.36}$$

is given by

$$\mathrm{Mon}(A)(\gamma) = \exp\left(-\oint_\gamma A\right)\,. \tag{2.37}$$

Note that the map (2.35) is far from being surjective in general, as one can see on purely dimensional grounds. Indeed, suppose C is compact and let g be the genus of C. Then,

- If $D = \emptyset$, $\dim \mathcal{M}^\circ_{\mathrm{dR}} = g$, while $\dim \mathcal{M}_{\mathrm{Betti}} = 2g$.
- If $|D| = n \neq 0$, $\dim \mathcal{M}^\circ_{\mathrm{dR}} = g + n - 1$, while $\dim \mathcal{M}_{\mathrm{Betti}} = 2g + n - 1$.

This tells us that for $g > 0$ "most" representations cannot be realized by connections with first-order poles on the trivial bundle. However, we know that they can all be realized by connections with first-order poles on *some* bundle according to Theorem 2.15.

Exercise 2.11 (Monodromy of rank 1 connections in the trivial bundle over a torus). Consider the case $C = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ and $D = \emptyset$. Then

$$\mathcal{M}_{\mathrm{Betti}}(\mathrm{GL}(1), C) \simeq (\mathbb{C}^\times)^2, \tag{2.38}$$

[12] In particular, if $g = e^\chi$ this becomes $A \sim A - \mathrm{d}\chi$.

i.e. a representation $\rho : \pi_1(C) \to \mathbb{C}^\times$ is determined by two holonomies $M_1, M_2 \in \mathbb{C}^\times$, as indicated in Figure 7.

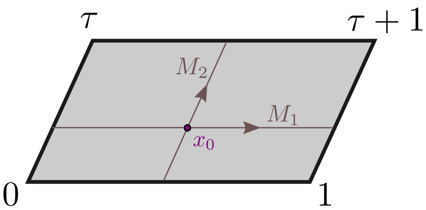

FIGURE 7. A fundamental domain for $C = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ and two cycles which generate $\pi_1(C)$.

Describe the image $\mathrm{Mon}(\mathcal{M}^\circ_{\mathrm{dR}}) \subset \mathcal{M}_{\mathrm{Betti}} \simeq (\mathbb{C}^\times)^2$, i.e. describe the set of representations which are realized by holomorphic connections on the trivial holomorphic bundle. Does $\mathrm{Mon}(\mathcal{M}^\circ_{\mathrm{dR}})$ depend on τ?

Exercise 2.12 (Constructing rank 1 connections with given monodromy). Continuing with the case $C = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ and $D = \emptyset$, suppose given some $M_1, M_2 \in \mathbb{C}^\times$. Describe directly a holomorphic rank 1 connection $(\mathcal{F}, \nabla)$ over C, whose monodromies are (M_1, M_2). There are various approaches, all equivalent, and it is interesting to explore the relations between them:

(1) Construct $\mathcal{F}$ as a quotient of the sheaf $\mathcal{O}$ on the universal cover $\widetilde{C} = \mathbb{C}$ (this follows our proof of Theorem 2.15).
(2) Construct $\mathcal{F}$ as the sheaf of C^∞ functions obeying $\{\partial_{\bar{z}}\psi = c\psi\}$ for a constant $c \in \mathbb{C}$.
(3) Construct $\mathcal{F}$ as a sheaf $\mathcal{O}([w] - [0])$ for some $w \in C$, i.e. a sheaf of meromorphic functions which are allowed a first-order pole at w and required to have a first-order zero at 0. (Then one direct way to construct connections ∇ on $\mathcal{F}$, and compute their monodromy, would involve learning a little about theta functions.)

2.6.2. *The genus zero case.* The case $C = \mathbb{CP}^1$ has naturally received a lot of attention. Hilbert's 21st problem asked:

> To show that there always exists a linear differential equation of the Fuchsian class, with given singular points and monodromic group.

Modern authors generally interpret "linear differential equation of the Fuchsian class" to mean a connection $(\mathcal{F}, \nabla)$ over $C = \mathbb{CP}^1$, with regular singularities, under the additional condition that the extension $\overline{\mathcal{F}}$ can be taken to be the trivial bundle $\mathcal{O}^N_{\mathbb{CP}^1}$.[13]

Now, the isomorphism classes of holomorphic bundles over $C = \mathbb{CP}^1$ are discrete, by the following:

Lemma 2.18 (Birkhoff-Grothendieck lemma on bundles over $\mathbb{CP}^1$). Fix $p \in \mathbb{CP}^1$ and a holomorphic rank N vector bundle $\mathcal{E}$ over $\mathbb{CP}^1$. Also fix a decomposition of the fiber $\mathcal{E}_{[p]}$ into lines, $\mathcal{E}_{[p]} = L_1 \oplus \cdots \oplus L_N$. Then there exist holomorphic line bundles $\mathcal{L}_i$ over $\mathbb{CP}^1$

[13] An alternative interpretation is that Hilbert meant a single higher-order ODE with regular singularities in the sense of Definition C.1, but this could not work for dimension-counting reasons, and apparently this was already known (due to Poincare) when Hilbert asked the question.

such that

$$\mathcal{E} = \bigoplus_{i=1}^{N} \mathcal{L}_i, \qquad (\mathcal{L}_i)_{[p]} = L_i. \tag{2.39}$$

Moreover, for each i there exists some $k_i \in \mathbb{Z}$ such that

$$\mathcal{L}_i \simeq \mathcal{O}_{\mathbb{CP}^1}(k_i[p]). \tag{2.40}$$

Proof. [give ref] □

Moreover, the trivial bundle $\mathcal{O}^N_{\mathbb{CP}^1}$ is semistable: in a family of holomorphic bundles over $\mathbb{CP}^1$, if one fiber is the trivial bundle, nearby fibers are also isomorphic to the trivial bundle. This leads us to expect that if some monodromy representation can be realized on the trivial bundle, then nearby ones also can be realized on the trivial bundle. This means that if we define $\mathcal{M}^\circ_{\mathrm{dR}}(G, \mathbb{CP}^1, D)$ to be the moduli space of connections for which $\overline{\mathcal{F}}$ can be taken to be trivial, we should expect

$$\dim \mathcal{M}^\circ_{\mathrm{dR}}(G, \mathbb{CP}^1, D) = \dim \mathcal{M}_{\mathrm{dR}}(G, \mathbb{CP}^1, D) = \dim \mathcal{M}_{\mathrm{Betti}}(G, S) \tag{2.41}$$

(unlike the higher genus case of §2.6.1, where $\dim \mathcal{M}^\circ_{\mathrm{dR}} < \dim \mathcal{M}_{\mathrm{dR}}$). We want to understand the monodromy map

$$\mathrm{Mon}^\circ : \mathcal{M}^\circ_{\mathrm{dR}}(G, \mathbb{CP}^1, D) \hookrightarrow \mathcal{M}_{\mathrm{Betti}}(G, S)\,. \tag{2.42}$$

Let's see how this looks concretely. Connections on the trivial bundle with first-order poles at $D = \{z_1, \dots, z_n\}$, all $z_i \neq \infty$, are of the form

$$\nabla_z = \partial_z - \sum_{i=1}^{n} \frac{m_i}{z - z_i}, \qquad m_i \in \mathfrak{gl}(N, \mathbb{C}), \quad \sum_{i=1}^{n} m_i = 0 \tag{2.43}$$

where the condition $\sum_i m_i = 0$ is needed in order to avoid a singularity at $z = \infty$. $\mathcal{M}^\circ_{\mathrm{dR}}(G, \mathbb{CP}^1, D)$ is the space of these connections up to meromorphic equivalence. Meromorphic equivalence includes in particular constant gauge transformations $g \in G$, which act by $m_i \mapsto g m_i g^{-1}$. There may also be some other discrete equivalences; for instance, if m_i and m_j are diagonalizable, there is a gauge transformation which shifts one (any) eigenvalue of m_i by $+1$ and simultaneously shifts one (any) eigenvalue of m_j by -1. These gauge transformations are called *Schlesinger transformations*.

Choose a basepoint $x_0 \in S$ and cycles generating $\pi_1(S; x_0)$, as in Figure 8.

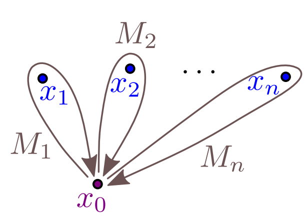

FIGURE 8. Cycles generating $\pi_1(S; x_0)$, labeled by their corresponding monodromy matrices M_i.

Then the monodromy map can be written[14]

$$\mathrm{Mon}^\circ_D : \{m = (m_1, \dots, m_n) : \sum_i m_i = 0\}/\sim \quad \hookrightarrow \quad \{M = (M_1, \dots, M_n) : \prod_i M_i = 1\}/\sim \tag{2.44}$$

where on both sides $\sim$ includes simultaneous conjugation by any $g \in \mathrm{GL}(N, \mathbb{C})$, and on the left side we have additional discrete equivalences. Note that the right side is a sort of nonlinear version of the left side, and in particular the dimensions on the two sides match, as predicted by (2.41).

We can understand Mon°_D as a fancy version of the exponential map. Indeed:

Exercise 2.13. Show that, in the special case where all the m_i commute, the map Mon°_D is given by $M_i = \exp(2\pi i m_i)$ (up to overall conjugation).

The map Mon°_D is surprisingly subtle. Here are a few results, mostly taken from the review in [5]:

- When $N = 1$, Mon°_D is surjective (since it is the exponential map).
- The image of Mon°_D contains all representations of $\pi_1(S)$ for which at least one of the M_i is diagonalizable (we'll prove this as Proposition 2.19 below).
- When $N = 2$, Mon°_D is surjective (by first choosing a meromorphic trivialization, then making brutal gauge transformations to eliminate higher-order poles [6]).
- The image of Mon°_D contains all irreducible representations of $\pi_1(S)$.
- When $N = 3$ and $n = 4$, or $N = 4$ and $n = 3$, Mon°_D is not surjective [7].[15]

The last result was a bit of a surprise historically: it was believed between 1908 and 1983 that Plemelj had proven Mon°_D is always surjective.

Proposition 2.19 (Monodromy map hits representations with one diagonalizable monodromy). If at least one M_i is diagonalizable, then $M = \mathrm{Mon}^\circ_D(m)$ for some m.

Proof. By Theorem 2.15 we know that M is the monodromy of some $(\mathcal{F}, \nabla)$ with regular singularities. We choose some extension $\overline{\mathcal{F}}$ where ∇ has first-order poles, and a point $p \in D$ for which the monodromy M is diagonalizable. Now Lemma 2.18 says that

$$\overline{\mathcal{F}} = \bigoplus_{i=1}^N \mathcal{L}_i \tag{2.46}$$

where each $\mathcal{L}_i \simeq \mathcal{O}(k_i[p])$ for some $k_i \in \mathbb{Z}$, and we can arrange this decomposition so that the residue $\mathrm{Res}_p \nabla \in \mathrm{End}(\overline{\mathcal{F}}_{[p]})$ preserves each $(\mathcal{L}_i)_{[p]}$. Then we consider a new bundle $\overline{\mathcal{F}}'$ related to $\overline{\mathcal{F}}$ by an elementary transformation: a section of $\overline{\mathcal{F}}'$ is a section of $\overline{\mathcal{F}}$, such

[14] We introduce the subscript D to emphasize that the map depends on the divisor D, even though we write the domain and codomain in a D-independent way.

[15] An explicit example of a representation not in the image of Mon°_D, for $N = 4$ and $n = 3$, is (quoted in [4]):

$$M_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 3 & 1 & 1 & -1 \\ -4 & -1 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -4 & -1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} -1 & 0 & 2 & -1 \\ 4 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & -1 \end{pmatrix}. \tag{2.45}$$

that the component in $\mathcal{L}_i$ has a zero of order at least k_i at p. By construction $\overline{\mathcal{F}}'$ is trivial, and ∇ still has first-order poles acting on $\overline{\mathcal{F}}'$. □

2.6.3. *Fixing orbits.* Continuing with the above example, suppose we fix adjoint orbits $\mathcal{O}_1, \ldots, \mathcal{O}_n \subset \mathfrak{g}$. We would like to consider a moduli space of connections for which the residues $m_i \in \mathcal{O}_i$.

This is a bit subtle, because as we have discussed there are meromorphic gauge transformations (Schlesinger transformations) which change the orbit where the m_i lie. It would be convenient to sidestep these by introducing a moduli space which is literally parameterized by the m_i up to overall conjugation, with no other equivalences. One way to achieve this is to make the story a bit more rigid, by working with connections in holomorphic bundles instead of meromorphic. In this way we define a new pair of moduli spaces $\mathcal{M}^{\circ}_{\mathrm{dR,p}}(G, \mathbb{CP}^1, D) \subset \mathcal{M}_{\mathrm{dR,p}}(G, \mathbb{CP}^1, D)$, and we can consider a subspace

$$\mathcal{M}^{\circ,\mathcal{O}}_{\mathrm{dR,p}}(G, \mathbb{CP}^1, D) \subset \mathcal{M}^{\circ}_{\mathrm{dR,p}}(G, \mathbb{CP}^1, D) \tag{2.47}$$

of connections where each $m_i \in \mathcal{O}_i$.

Similarly, given conjugacy classes $\mathcal{O}^{\times}_1, \ldots, \mathcal{O}^{\times}_n \subset G$ we can define a subspace

$$\mathcal{M}^{\mathcal{O}^{\times}}_{\mathrm{Betti}}(G, S) \subset \mathcal{M}_{\mathrm{Betti}}(G, S) \tag{2.48}$$

of connections where each $M_i \in \mathcal{O}^{\times}_i$.

The simplest case is the case when each $\mathcal{O}_i$ is a regular semisimple non-resonant orbit (i.e. it consists of diagonalizable matrices, with no pair of eigenvalues differing by an integer.) Let us restrict to that case, and suppose $\mathcal{O}^{\times}_i = \exp(2\pi \mathrm{i} \mathcal{O}_i)$. Then using Exercise 2.4, we can restrict Mon_D to a map

$$\mathrm{Mon}^{\mathcal{O}}_D : \mathcal{M}^{\circ,\mathcal{O}}_{\mathrm{dR,p}}(G, \mathbb{CP}^1, D) \to \mathcal{M}^{\mathcal{O}^{\times}}_{\mathrm{Betti}}(G, S), \tag{2.49}$$

concretely

$$\mathrm{Mon}^{\mathcal{O}}_D : \{m \in \prod \mathcal{O}_i : \sum m_i = 0\}/\sim \quad \to \quad \{M \in \prod \mathcal{O}^{\times}_i : \prod M_i = 1\}/\sim \tag{2.50}$$

Let us count dimensions: each orbit $\mathcal{O}_i$ has dimension $N^2 - N$; the condition $\sum m_i = 0$ reduces the dimension by $N^2 - 1$,[16] and the conjugation reduces the dimension again by $N^2 - 1$,[17] so the total dimension is

$$\dim \mathcal{M}^{\circ,\mathcal{O}}_{\mathrm{dR,p}}(G, \mathbb{CP}^1, D) = n(N^2 - N) - 2(N^2 - 1)\,. \tag{2.51}$$

In particular, for $N = 2$ this becomes

$$\dim \mathcal{M}^{\circ,\mathcal{O}}_{\mathrm{dR,p}}(G, \mathbb{CP}^1, D) = 2n - 6. \tag{2.52}$$

Incidentally, in general it is not so easy to see whether $\mathcal{M}^{\circ,\mathcal{O}}_{\mathrm{dR,p}}$ or $\mathcal{M}^{\mathcal{O}^{\times}}_{\mathrm{Betti}}$ is actually nonempty, for a given choice of $\mathcal{O}$ or $\mathcal{O}^{\times}$: this is called the (additive or multiplicative) *Deligne-Simpson problem*. It is a pure linear algebra problem. For the multiplicative version some progress was made in [8], using the relation between connections and Higgs bundles. The additive version was solved in [9] by identifying $\mathcal{M}^{\circ,\mathcal{O}}_{\mathrm{dR,p}}$ as a quiver variety; using the

[16] The -1 arises because the condition $\sum \mathrm{Tr}\, m_i = 0$ is already enforced by the choice of orbits $\mathcal{O}_i$.

[17] This time the -1 comes from the 1-dimensional center of G.

Riemann-Hilbert map this gives some information about the multiplicative version too. But never mind this; our statements will be true whether or not the spaces involved are empty.

2.6.4. *Isomonodromic deformation.* The map Mon_D (or $\mathrm{Mon}_D^{\mathcal{O}}$) depends on D. One can ask: if we vary D while holding the monodromies M constant, how does the corresponding $m = \mathrm{Mon}_D^{-1}(M)$ vary? This is the problem of *isomonodromic deformation*. Up to gauge equivalence, the answer is provided by the *Schlesinger equations*,

$$\mathrm{d}m_i = \sum_{j \neq i} [m_i, m_j] \, \mathrm{d} \log(z_i - z_j) \tag{2.53}$$

or equivalently

$$\partial_{z_i} m_i = \sum_{j \neq i} \frac{[m_i, m_j]}{z_i - z_j}, \qquad \partial_{z_j} m_i = \frac{[m_j, m_i]}{z_i - z_j} \quad (i \neq j) \tag{2.54}$$

Exercise 2.14. Prove that, if the m_i are functions of the z_i obeying (2.54), the monodromy $M = \mathrm{Mon}_D(m)$ is independent of the z_i. (Hint: consider a *system* of differential equations, i.e. a flat connection over the parameter-space $\{(z, z_1, \ldots, z_n) : z \neq z_i, z_i \neq z_j\}$, where the connection in the z direction is ∇_z given in (2.43), and the equation in the z_i direction is

$$\nabla_{z_i} = \partial_{z_i} + \frac{m_i}{z - z_i}. \tag{2.55}$$

Check that the integrability condition for this system, $[\nabla_z, \nabla_{z_i}] = 0$, is equivalent to the equations (2.54). Using the Frobenius integrability theorem, this integrability implies the solutions $\Psi(z, z_1, \ldots, z_n)$ form a local system of rank N, now over the higher-dimensional parameter space $(z, z_1, \ldots, z_n)$. Show that this in turn implies the desired z_i-invariance of the monodromy.)

Even forgetting their origin in the isomonodromy problem, these equations are interesting in themselves, as an integrable system of nonlinear PDE.

Example 2.20 (Painlevé VI). The first interesting case is $N = 2$, $n = 4$. In that case, according to our dimension count (2.51), both sides of the monodromy map $\mathrm{Mon}_D^{\mathcal{O}}$ have dimension 2.

We follow the account in [10]. By a Möbius transformation we may as well assume the z_i are $(0, t, 1, \infty)$. Thus we have a single parameter t which varies on the twice-punctured plane $\mathbb{C} \setminus \{0, 1\}$, and a family of 2-dimensional complex spaces $\mathcal{M}_{\mathrm{dR,p}}^{\mathcal{O}}(t)$, equipped with a flat Ehresmann connection given by isomonodromy.

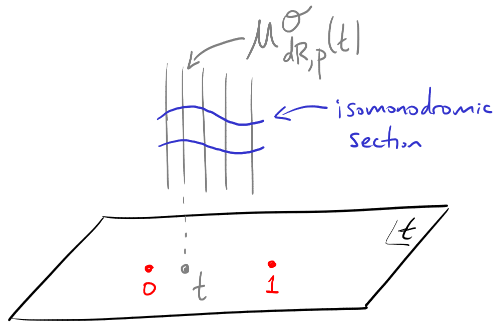

FIGURE 9. The bundle of complex surfaces $\mathcal{M}^{\mathcal{O}}_{\mathrm{dR,p}}(t)$ over $\mathbb{C} \setminus \{0, 1\}$ and its flat isomonodromy connection.

This Ehresmann connection is given by (2.54), which in this case becomes

$$\partial_t m_1 = \frac{[m_2, m_1]}{t}, \qquad \partial_t m_2 = \frac{[m_1, m_2]}{t} + \frac{[m_3, m_2]}{t-1}, \qquad \partial_t m_3 = \frac{[m_2, m_3]}{t-1} \tag{2.56}$$

To describe it more concretely we introduce a coordinate system (x, y) on $\mathcal{M}^{\circ,\mathcal{O}}_{\mathrm{dR,p}}(t)$, as follows. Let us choose the $\mathcal{O}_i$ so that all the m_i are traceless, and let $m_4 = -(m_1 + m_2 + m_3) = -\lim_{z\to\infty} z A_z(z)$. Then let $\pm\frac{1}{2}\theta_i$ be the eigenvalues of m_i, and conjugate the m_i so that $m_4 = \mathrm{diag}(\frac{1}{2}\theta_4, -\frac{1}{2}\theta_4)$. Then A_z must have the form

$$A_z(z) = -\frac{1}{z(z-1)(z-t)} \begin{pmatrix} \frac{1}{2}z^2\theta_4 + P(z) & Q(z) \\ R(z) & -\frac{1}{2}z^2\theta_4 - P(z) \end{pmatrix} \tag{2.57}$$

where P, Q, R are all polynomials with degree ≤ 1. Our coordinate system (x, y) will cover only the locus where Q has degree exactly 1. In that case, let y be the unique zero of Q, and x be the top left entry of $A_z(z = y)$, i.e. $x = \frac{1}{2}y^2\theta_4 + P(y)$.[18]

Then (2.56) implies a first-order nonlinear coupled ODE for $(x(t), y(t))$, and eliminating $x(t)$ it becomes a second-order nonlinear ODE for $y(t)$:

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{1}{2}\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} \\ + \frac{y(y-1)(y-t)}{2t^2(t-1)^2}\left((\theta_4 - 1)^2 - \frac{\theta_1^2 t}{y^2} + \frac{\theta_3^2(t-1)}{(y-1)^2} + \frac{(1-\theta_2^2)t(t-1)}{(y-t)^2}\right). \end{aligned} \tag{2.58}$$

This equation is famous: it is the sixth (and most general) on the list of second-order ODEs with the "Painlevé property," namely that for a general solution $y(t)$ the only movable singularities are poles.

Let us say a bit more about this. By a "movable" singularity we mean one whose position depends on the particular solution we consider. The phenomenon of movable singularities doesn't occur for linear equations: there the only singularities occur at the singularities of the coefficients of the equation. For nonlinear equations, in contrast, solutions can blow up even at a generic point. The Painlevé property of (2.58) is that when such blowing-up occurs it is only a pole of $y(t)$ rather than something worse. In particular, any local

[18] Note that both of these quantities are invariant under conjugation of A_z by a diagonal matrix, as they must be since we did not fix that freedom.

solution $y(t)$ of (2.58) has meromorphic continuation along an arbitrary path, suffering from monodromy only around the fixed singularities, which are at $t = 0, 1, \infty$. This property should be understood as coming from the geometric interpretation of (2.58) as a flat Ehresmann connection over $\mathbb{C} \setminus \{0, 1\}$.

This picture is reviewed in [11] which also gives a more detailed picture of the moduli space $\mathcal{M}^{\mathcal{O}}_{\mathrm{dR,p}}$ appearing here: $\mathcal{M}^{\mathcal{O}}_{\mathrm{dR,p}} = X \setminus D$ where X is a certain rational surface (blow-up of a Hirzebruch surface $\mathbb{F}_2$ at 8 points), and D is an effective anticanonical divisor with five components arranged in an affine D_4 configuration.

3. Connections with irregular singularities

We continue considering meromorphic rank N connections on a Riemann surface C, but now drop the condition of regular singularities.

We will consider an "irregular curve" in the language of Boalch (e.g. [12]): this means C with marked points D, together with additional data Θ_p for each $p \in D$. Each Θ_p captures the exponential behavior of flat sections of a meromorphic connection near the marked point p.

Given (C, D, Θ) we are going to define two categories. First, we will have $\mathcal{C}_{\mathrm{dR}}(G, C, D, \Theta)$, the category of meromorphic connections with irregular singularity controlled by Θ. Second, we will replace S by its real blow-up $\widehat{S}_D$, extract a topological object θ from Θ, and define a category of Stokes graded local systems, $\mathcal{C}_{\mathrm{Betti}}(G, \widehat{S}_D, \theta)$.

3.1. Asymptotic exponents.

Fix $p \in D$ and introduce a local coordinate z around p. We will work in coordinate systems where p may be either $z = 0$ or $z = \infty$.

When we consider flat sections of connections with irregular singularity at p, there is a characteristic exponential growth or decay, determined by the singular terms in the connection. For example, for the Airy equation we have seen that there are solutions behaving like $\exp(\pm \frac{2}{3} z^{\frac{3}{2}})$. More generally, we can meet solutions behaving like $\exp(q)$ with $q = \sum_\alpha c_\alpha z^\alpha$.

The real parts of the exponents q will play a key role in the story. In simple cases (e.g. when all exponents are monomials) one can quickly capture their behavior by a *Stokes diagram* as indicated below.

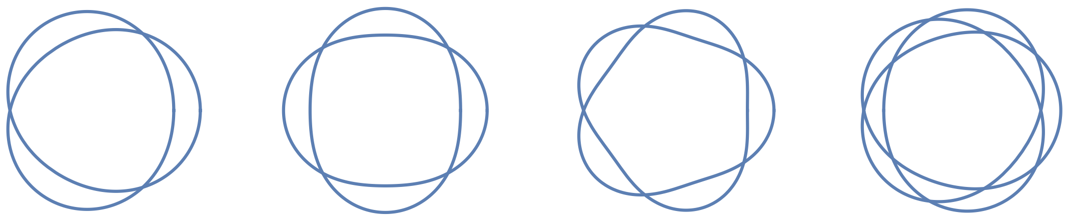

FIGURE 10. Stokes diagrams corresponding to the exponent-sets $\{\pm z^{3/2}\}$, $\{\pm z^2\}$, $\{\pm z^{5/2}\}$, $\{e^{2\pi i k/3} z^{5/3}\}$. The diagram is a polar plot in (r, ϕ) of the locus $r = 1 + \epsilon \operatorname{Re}(q(m e^{i\phi}))$, for a suitably chosen ϵ and m; for this figure I took $\epsilon = 0.15$ and $m = 1$. Thus, for any given phase ϕ, the points which are further from the origin represent the exponents with larger value of $\operatorname{Re}(q)$; crossing points occur at phases ϕ where two exponents have the same $\operatorname{Re}(q)$. If the exponents q are related to a differential equation with irregular singularity, the crossing points occur at anti-Stokes rays, while the points where two exponents are maximally separated occur at Stokes rays.

We want to formalize this structure. We let $\widehat{C}_D$ denote the real blow-up of C at the points of D (see §B.5), and let $\partial_p \subset \widehat{C}_D$ denote the boundary circle associated to $p \in D$. Any subset $Z \subset \partial_p$ can be thickened to a sector (formal or actual) $\tilde{Z} \subset \widehat{C}_D$.

Now we define, essentially following [12]:

Definition 3.1 (Local system of all possible asymptotic exponents). The local system $\mathcal{E}_p$ of all possible asymptotic exponents is a local system of sets over ∂_p, whose sections on $U \subset \partial_p$ are finite sums

$$q = \sum_{\alpha} c_\alpha z^\alpha, \qquad c_\alpha \in \mathbb{C}, \alpha \in \mathbb{Q}, \tag{3.1}$$

with all terms growing as $z \to p$ (so $\alpha < 0$ if $p = 0$, $\alpha > 0$ if $p = \infty$). Here the notation z^α means any branch of the multivalued function z^α on a small thickening $\tilde{U} \subset \widehat{C}_p$.

We remark that the local system $\mathcal{E}_p$ does not depend on the choice of the local coordinate z around p, in the following sense. We regard $q \in \mathcal{E}_p$ as giving the singular part of a function $q(z)$, i.e. we consider q modulo functions which tend to 0 as $z \to p$. The set of such singular parts is independent of the choice of coordinate (although q represented by a monomial in one coordinate might be a polynomial in another coordinate.)

Definition 3.2 (System of asymptotic exponents). A *system of asymptotic exponents at p* is a local system of finite sets Θ_p over ∂_p, equipped with a map $q : \Theta_p \to \mathcal{E}_p$.

When we discuss only a single point p we sometimes drop the explicit subscript p (especially when we need to make room for other subscripts as we do below).

As for any local system of sets, we can view Θ as a covering space of ∂. Each sheet of Θ is labeled by an asymptotic exponent q, in a way compatible with the parallel transport in $\mathcal{E}$. The map $q : \Theta \to \mathcal{E}$ gives an equivalence relation on Θ; let $\overline{\Theta}$ be the quotient by this relation. $\overline{\Theta}$ is a sub-local system of $\mathcal{E}$, with connected components $\Theta_1, \ldots, \Theta_\ell$ labeled by multiplicities $K_1, \ldots, K_\ell \in \mathbb{N}$, and

$$|\Theta| = \sum_{i=1}^{\ell} K_j |\Theta_j| \,. \tag{3.2}$$

In many important examples no exponent is repeated, so then the equivalence relation is trivial, all multiplicities $K_j = 1$, and $\overline{\Theta} = \Theta$. In these cases we will just write Θ directly as a subsystem of $\mathcal{E}$.

Example 3.3 (Asymptotic exponents for the Airy equation). We can take $p = \infty$ and let $\Theta \subset \mathcal{E}$ be

$$\Theta = \left\{ \pm \frac{2}{3} z^{3/2} \right\} . \tag{3.3}$$

In this case $|\Theta| = 2$, and monodromy around ∂ exchanges the two sheets. Thus Θ has one connected component, with $K = 1$.

As one might suspect from our preliminary discussion above, the system Θ is canonically associated to the Airy equation (1.12): see Example 3.11 below.

Example 3.4 (Asymptotic exponents for Schrödinger equations with polynomial potential). More generally, will see below that, in the example of a Schrödinger equation (1.38) with monic polynomial potential of degree n, we naturally obtain a system $\Theta \subset \mathcal{E}$ consisting of two sections

$$\Theta = \left\{ \pm \left(\frac{2}{n+2} z^{(n+2)/2} + B(z) \right) \right\} . \tag{3.4}$$

Here $B(z)$ is z times a polynomial of degree $\leq (n-2)/2$ for n even, or $z^{\frac{1}{2}}$ times a polynomial of degree $\leq (n-1)/2$ for n odd. Again here $|\Theta| = 2$. Monodromy around ∂ exchanges the two sections when n is odd, or acts trivially when n is even.

In particular, when $n = 3$ this becomes

$$\Theta = \left\{ \pm \left(\frac{2}{5} z^{5/2} + a_1 z^{3/2} + a_2 z^{1/2} \right) \right\} . \tag{3.5}$$

[restore comments about change of variable shifting away top coefficient somewhere?]

Example 3.5 (Asymptotic exponents for connections with regular singularity). We can also consider the opposite extreme, where Θ consists of the trivial section 0, with multiplicity N. In other words, Θ is a constant local system modeled on $\underline{N}$, with the constant map $q : \Theta \to \mathcal{E}$ taking everything to 0. We can write this as

$$\Theta = \{0, 0, \ldots, 0\}. \tag{3.6}$$

Thus $|\Theta| = N$ and $\overline{\Theta} \to \partial$ is the trivial 1-sheeted covering.

This example arises in the case of a connection in a bundle of rank N with a regular singularity at p: in that case, as we have seen, the solutions have only moderate growth, or said otherwise, they have exponential growth with exponent $q = 0$.

3.2. **Model connections with fixed asymptotic exponents.** We want to discuss meromorphic connections whose asymptotics are controlled by a system of asymptotic exponents Θ. We begin by considering a local model for what such a connection could look like.

Definition 3.6 (Local model for a meromorphic connection). Suppose given a system of asymptotic exponents Θ at p, with connected components $\Theta_j \subset \overline{\Theta}$, occurring with multiplicities K_j. Then, given matrices $m_j \in \mathfrak{gl}(K_j, \mathbb{C})$, we define a meromorphic connection $(\mathcal{F}^\Theta, \nabla^{\Theta,m})$ in a neighborhood of p, as follows.

For each j, we consider a meromorphic connection $(\mathcal{F}^{\Theta_j}, \nabla^{\Theta_j,m})$ over a thickening $\tilde{\Theta}_j$ of Θ_j. To write it explicitly we introduce a local coordinate $\zeta = z^{1/|\Theta_j|}$ on $\tilde{\Theta}_j$. Then

$$\mathcal{F}^{\Theta_j} = \mathcal{O}^{\oplus K_j}, \qquad \nabla_\zeta^{\Theta_j,m_j} = \partial_\zeta - \partial_\zeta q - \frac{m_j}{\zeta}. \tag{3.7}$$

Next we push each $(\mathcal{F}^{\Theta_j}, \nabla^{\Theta_j,m})$ forward to C, and take the direct sum over j, to get a meromorphic connection $(\mathcal{F}^\Theta, \nabla^{\Theta,m})$ in a neighborhood of p.

Example 3.7 (Model form with all multiplicities 1). We consider the case where all multiplicities are 1; in this case the model form amounts to a simple explicit connection in a line bundle over the covering Θ. Θ consists of connected components $\Theta_1, \ldots, \Theta_\ell$, with

$$\sum_{j=1}^{\ell} |\Theta_j| = |\Theta| = N, \tag{3.8}$$

and the parameters m reduce to complex numbers

$$m_1, \ldots, m_\ell \in \mathbb{C}. \tag{3.9}$$

Upon choosing a local isomorphism $\Theta \to \underline{N}$ we get a local isomorphism $\mathcal{F}^\Theta \simeq \mathcal{O}_C^N$ around p, and then

$$\nabla_z^{\Theta,m} = \partial_z - \operatorname{diag}\left(\partial_z q_i + \frac{m_i'}{z}\right) \tag{3.10}$$

where each $m_i' = m_j/|\Theta_j|$ for some j.[19]

Note that the local model connection $(\mathcal{F}^\Theta, \nabla^{\Theta,m})$ has a basis of local solutions of the form

$$\psi^{(i)} = \left(0, \ldots, \mathrm{e}^{q_i} z^{m_i'}, \ldots, 0\right). \tag{3.11}$$

This motivates the interpretation of q_i as asymptotic exponents of solutions.

Example 3.8 (Model form with all multiplicities 1 and a single component). We further specialize to the case where Θ has just one connected component. In this case we just have a single parameter $m \in \mathbb{C}$, and locally where Θ is trivialized we can write

$$\nabla_z^{\Theta,m} = \partial_z - \operatorname{diag}\left(\partial_z q_i + \frac{m'}{z}\right) \tag{3.12}$$

where $m' = m/N$. Note that in this case all the q_i are related to one another by analytic continuation around ∂.

[19] The fastest way to understand the factor of $|\Theta_j|$ appearing here is to think about the connection 1-forms: if $\zeta = z^p$, then $\mathrm{d}\zeta/\zeta = p\,\mathrm{d}z/z$.

Example 3.9 (Local model for the Airy equation). We consider the system of exponents $\Theta = \{\pm\frac{2}{3}z^{3/2}\}$ around $z = \infty$ (Example 3.3). Then, on the complement of a branch cut for $z^{1/2}$, the local model connection is

$$\nabla_z^{\Theta,m} = \partial_z - \operatorname{diag}\left(z^{1/2} + \frac{m}{2z}, -z^{1/2} + \frac{m}{2z}\right). \tag{3.13}$$

In Example 3.11 below we will see that this connection, with $m = \frac{1}{2}$, indeed arises as a formal model for the Airy equation.

Example 3.10 (Local model for all exponents zero). We can also consider the case of Example 3.5, where $q = 0$, so Θ just has one component, of multiplicity N. In that case we have

$$\nabla_z^{\Theta,m} = \partial_z - \frac{m}{z} \tag{3.14}$$

for $m \in \mathfrak{gl}(N,\mathbb{C})$. Thus we have recovered our standard model for connections with regular singularity.

3.3. **Meromorphic versus formal equivalence.** Now, we could consider connections which are meromorphically equivalent to the model form $(\mathcal{F}^\Theta, \nabla^{\Theta,m})$. It turns out that this condition is in general too strong to impose, as we see in the following example.

Example 3.11 (Formally diagonalizing the Airy equation). We consider the Airy equation, i.e. the connection $(\mathcal{O}^2, \nabla)$ with

$$\nabla_z = \partial_z + \begin{pmatrix} 0 & -1 \\ -z & 0 \end{pmatrix}. \tag{3.15}$$

The most direct approach to putting this connection in the model form would be to make a gauge transformation by a matrix g_0 which diagonalizes it, for instance[20]

$$g_0 = \begin{pmatrix} \sqrt{z} & 1 \\ -\sqrt{z} & 1 \end{pmatrix}. \tag{3.16}$$

Making this transformation we obtain

$$\nabla'_z = \partial_z + g_0 A_z g_0^{-1} + g_0 \partial_z g_0^{-1} = \partial_z + \begin{pmatrix} -\sqrt{z} & 0 \\ 0 & \sqrt{z} \end{pmatrix} + \frac{1}{4z}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{4z}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.17}$$

The first three terms match $\nabla^{\Theta,m}$ with $m = \frac{1}{2}$, but the final, off-diagonal, term is unwanted.

One can now attempt to eliminate this unwanted off-diagonal term as follows. The key fact is that the eigenvalues of the leading part of A'_z are distinct, and thus using the formula

$$\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & -2a \\ 2b & 0 \end{pmatrix} \tag{3.18}$$

[20] You might worry that g_0 as written here involves $\sqrt{z}$, so it does not on its face look like a meromorphic gauge transformation of the kind we usually consider. But g_0 represents a map $\mathcal{O}_C^2 \to \mathcal{F}^\Theta$, written relative to a *local* trivialization of $\mathcal{F}^\Theta$, coming from a local trivialization $\Theta \to \underline{2}$ off a branch cut of $\sqrt{z}$. With this understanding, the necessary condition for g_0 to be well defined is that it should transform by $g_0 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g_0$ on crossing the cut, which indeed it does.

we see that off-diagonal matrices are in the image of $[A'_z, \cdot]$ to leading order in z. We will make successive gauge transformations by matrices $g_1, g_2, \ldots$, where[21]

$$g_n = \mathbf{1} + X_n, \quad X_n = z^{-\frac{3}{2}n} \begin{pmatrix} 0 & a_n \\ b_n & 0 \end{pmatrix}, \qquad a_n = (-1)^n b_n. \tag{3.19}$$

To leading order as $z \to \infty$, we have $g_n^{-1} = \mathbf{1} - X_n$, and

$$g_n A'_z g_n^{-1} + g_n \partial_z g_n^{-1} - A'_z = [X_n, A'_z] + O(z^{-\frac{3}{2}n-1}) = z^{-\frac{3}{2}n+\frac{1}{2}} \begin{pmatrix} 0 & -2a_n \\ 2b_n & 0 \end{pmatrix} + O(z^{-\frac{3}{2}n-1}) \tag{3.20}$$

Thus by choosing the coefficients a_n, b_n appropriately we can eliminate the unwanted term at order $z^{-\frac{3}{2}n+\frac{1}{2}}$. For instance, to cancel the off-diagonal term with coefficient $\frac{1}{4z}$ in (3.17) we need $a_1 = -\frac{1}{8}$, $b_1 = \frac{1}{8}$. Continuing in this way, we find coefficients

$$(b_n)_{n=1}^{\infty} = \frac{1}{8}, \frac{3}{32}, \frac{71}{512}, \cdots \qquad a_n = (-1)^n b_n \tag{3.21}$$

After the transformation by g_n the remaining off-diagonal part is of order $z^{-\frac{3}{2}n-1}$.

This process generates additional diagonal terms, e.g. the 11 entry of the resulting connection A'_z becomes

$$-\sqrt{z} - \frac{1}{4z} + \left(-\frac{1}{32} z^{-\frac{5}{2}} + \frac{3}{128} z^{-4} + \cdots \right) \tag{3.22}$$

but the terms in parentheses here can be written as $\mathrm{d} \log f$ for a formal power series $f = 1 + \frac{1}{48} z^{-\frac{3}{2}} + \cdots$, and thus removed by a further diagonal gauge transformation.

Composing all these gauge transformations we obtain the desired g as a formal power series in $z^{-\frac{3}{2}}$, and conclude that the connection $(\mathcal{O}^2, \nabla)$ is formally equivalent to $(\mathcal{F}^\Theta, \nabla^{\Theta,m})$ with $m = \frac{1}{2}$.

However, this series cannot converge to an honest meromorphic gauge transformation g. To see this, imagine that it did converge, so that $(\mathcal{O}^2, \nabla)$ would be meromorphically equivalent to $(\mathcal{F}^\Theta, \nabla^{\Theta,m})$ in some neighborhood of $z = \infty$. In that case its monodromy around a loop around $z = \infty$ would have to match that of $(\mathcal{F}^\Theta, \nabla^{\Theta,m})$. But $(\mathcal{O}^2, \nabla)$ has trivial monodromy around such a loop, while that of $(\mathcal{F}^\Theta, \nabla^{\Theta,m})$ is nontrivial by Exercise 3.1 below. This gives the desired contradiction.

Exercise 3.1 (Formal monodromy of the Airy equation). Show that the monodromy of $(\mathcal{F}^\Theta, \nabla^{\Theta,m})$ in this example, relative to the basis coming from an identification $\Theta \to \underline{2}$, is $M = \begin{pmatrix} 0 & \mathrm{i} e^{\frac{4}{3} z^{\frac{3}{2}}} \\ \mathrm{i} e^{-\frac{4}{3} z^{\frac{3}{2}}} & 0 \end{pmatrix}$. (Note that this coefficient ensures that the monodromy has determinant 1; see the next exercise.) [check signs]

Exercise 3.2.

[21] For $n > 0$, g_n represents a map $\mathcal{F}^\Theta \to \mathcal{F}^\Theta$, again relative to a local trivialization of Θ off a branch cut. Thus the necessary condition is that it should transform by $g_n \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on crossing the cut, which it does.

- Suppose $(\mathcal{F}, \nabla)$ and $(\mathcal{F}', \nabla')$ are two connections of rank $N = 1$ with irregular singularity at p, and that they are formally equivalent around p. Show that they are meromorphically equivalent in a neighborhood of p.
- Suppose $(\mathcal{F}, \nabla)$ and $(\mathcal{F}', \nabla')$ are two connections with irregular singularity at p, and that they are formally equivalent around p. Show that the monodromies M, M' of the two connections have $\det M = \det M'$.

The key lesson of Example 3.11 is that for irregular singularities there is a difference between meromorphic equivalence and formal equivalence. This difference is important: it is at the root of the Stokes phenomenon for these singularities. Here is another example which makes the same point:

Exercise 3.3. Consider a connection in $\mathcal{O}^2$ over $C = \mathbb{CP}^1$ of the form $\nabla_z = \partial_z + \frac{u}{z^2} - \frac{A}{z}$, with $u, A \in \mathfrak{gl}(2, \mathbb{C})$. Assume u is diagonal with distinct entries, but A is not necessarily diagonal. This connection has singularities at $z = 0$ and $z = \infty$. Show that around $z = 0$ ∇ is formally equivalent to ∇' given by $\nabla'_z = \partial_z + \frac{u}{z^2} - \frac{m}{z}$ for some diagonal matrix m, but in general m is not conjugate to A. Conclude that in general the equivalence is only formal.

3.4. Category of irregular connections.

Every meromorphic connection is formally equivalent to a unique model:

Proposition 3.12 (Every meromorphic connection has some asymptotic exponents). Suppose $(\mathcal{F}, \nabla)$ is a meromorphic connection over C with poles at D. Then for each $p \in D$ there is a unique system of asymptotic exponents Θ_p such that $(\mathcal{F}, \nabla)$ is formally equivalent to some $(\mathcal{F}^{\Theta_p}, \nabla^{\Theta_p, m})$ around p.

Proof. This is often called the Hukuhara-Turritin Theorem. The construction of the desired formal equivalence is essentially a systematic elaboration of what we did for the Airy equation in Example 3.11. It is somewhat technical and I will not attempt to reproduce it here, but it is discussed in many places. For example, it is (after some mild translation) part 3 of Theorem 3.1 in [13]. There is also a nice (and succinct!) account at [14]. □

When $(\mathcal{F}, \nabla)$ is formally equivalent to $(\mathcal{F}^{\Theta_p}, \nabla^{\Theta_p, m})$ around p, we say that $(\mathcal{F}, \nabla)$ *has asymptotic exponents* Θ and *exponent of formal monodromy* m.

Exercise 3.4 (Asymptotic exponents for Schrödinger equations with polynomial potential). Carry out the formal diagonalization procedure for the Schrödinger equation with polynomial potential, (1.38). Show that the asymptotic exponent $q(z)$ is the singular part of $\pm \int^z \sqrt{-P(z')}\, \mathrm{d}z'$.

With Proposition 3.12 as motivation, we define a category of connections with irregular singularities and fixed asymptotic exponents:

Definition 3.13 (de Rham category with irregular singularities). Let $\mathcal{C}_{\mathrm{dR}}(G, C, D, \Theta)$ be the category whose objects are meromorphic connections $(\mathcal{F}, \nabla)$ over C with poles at D and asymptotic exponents Θ, and whose morphisms are meromorphic isomorphisms of meromorphic connections.

3.5. Stokes and anti-Stokes rays.

In our discussion of the Airy function in §1.2, it was important that in a generic direction $\phi \in \partial$ one has a dominance relation among the exponentials occurring in Θ_ϕ: either $\mathrm{Re}(z^{\frac{3}{2}}) \gg \mathrm{Re}(-z^{\frac{3}{2}})$ or $\mathrm{Re}(z^{\frac{3}{2}}) \ll \mathrm{Re}(-z^{\frac{3}{2}})$ as $z \to \infty$ in the direction ϕ. In this section we formalize part of this structure, again mostly following the notation of [12].

Definition 3.14 (Dominance relations). We fix a point p. Suppose given a system of asymptotic exponents Θ around p, and any $\phi \in \partial_p$. Let $(q - q')_l$ be the leading term in $q - q'$ as $z \to p$ along ϕ. Then we have:

- A partial ordering $<$ on Θ_ϕ, defined by: $q < q'$ if $\mathrm{Re}(q - q')_l < 0$.
- A partial ordering $\prec$ on Θ_ϕ, defined by: $q \prec q'$ if $\mathrm{Re}(q - q')_l$ is minimized at ϕ. (Equivalently, if $\mathrm{Re}(q - q')_l < 0$ and $\mathrm{Im}(q - q')_l = 0$.)

So $q < q'$ if $e^q \ll e^{q'}$ as $z \to p$, and $q \prec q'$ along the rays where this decay occurs fastest. The ordering $<$ is generically a total ordering, while $\prec$ is generically trivial.

Definition 3.15 (Stokes and anti-Stokes rays). Given a system of asymptotic exponents Θ,

- A *Stokes ray* is a $\phi \in \partial$ such that $\prec$ is not trivial on Θ_ϕ.
- An *anti-Stokes ray* is a $\phi \in \partial$ such that $<$ is not a total order on Θ_ϕ.

In our treatment the Stokes rays will be the main players. Here are the Stokes rays in a few simple cases.

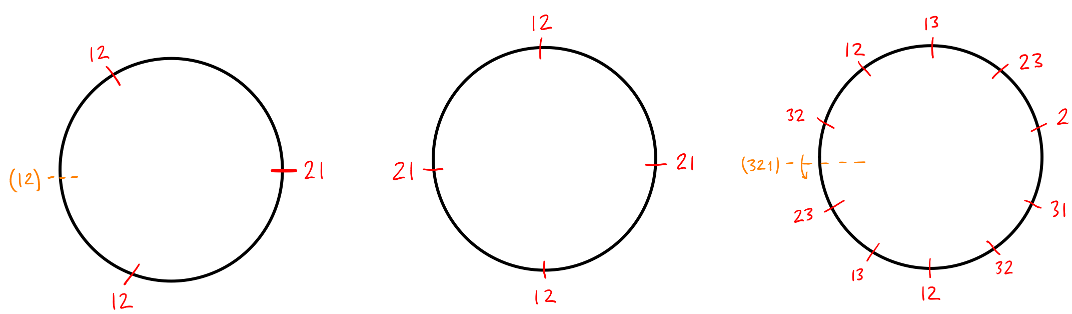

FIGURE 11. Patterns of Stokes rays for the exponent systems around $z = \infty$ given by $\{q^2 = z^3\}$, $\{q^2 = z^4\}$, $\{q^3 = z^5\}$. In each case we have trivialized the covering $\Theta \to \underline{N}$ by choosing the standard N-th roots, away from a branch cut (dashed orange), and we represent the relation $i \prec j$ at a Stokes ray by the condensed notation ij.

Exercise 3.5. Verify the patterns of Stokes rays shown in Figure 11.

Exercise 3.6. Verify the following, in the examples above:

- Changing the asymptotic exponents by adding subleading terms does not affect the Stokes rays.
- Multiplying all the asymptotic exponents by an overall coefficient has the effect of rotating the whole pattern of Stokes rays.

Exercise 3.7. Draw the patterns of Stokes rays for the exponent systems around $z = \infty$ given by $\{q^3 = z^6\}$ and $\{q^4 = z^5\}$. (Note the latter example is less symmetrical than the ones we have looked at so far, roughly because the differences of 4th roots of unity are not all related by monodromy.)

3.6. Stokes graded local systems.

Again we are roughly following [12].

Definition 3.16 (I-gradings). Suppose V is a vector space and I a multiset, i.e. a set with a function $K : I \to \mathbb{N}$ ("multiplicity"). An *I-grading* of V is a subspace $G^{(i)}$ for each $i \in I$, such that

$$V = \bigoplus_i G^{(i)}, \qquad \dim G^{(i)} = K_i. \tag{3.23}$$

Definition 3.17 (Stokes condition). Suppose V is a vector space and I a multiset as above. Suppose I is equipped with a partial order $\prec$, and G, G' are two I-gradings of V. We define

$$G^{(\preceq j)} = \bigoplus_{i:i\preceq j} G^{(i)}. \tag{3.24}$$

We say G, G' obey the *Stokes condition* if for all $j \in I$

$$G'^{(\preceq j)} = G^{(\preceq j)}. \tag{3.25}$$

Note that if $\prec$ is trivial the Stokes condition simply says $G = G'$.

Example 3.18 (Stokes condition in generic case). Suppose there is just a single pair $i \prec j$. In this case the meaning of the Stokes condition is

$$G'^{(k)} = G^{(k)} \text{ for } k \neq j, \qquad G'^{(j)} \oplus G^{(i)} = G^{(j)} \oplus G^{(i)}. \tag{3.26}$$

Thus in going from G to G' the only piece of the grading that changes is $G^{(j)}$, and it changes only by mixing with $G^{(i)}$.

When $|I| = 2$ this condition reduces further, to

$$G^{(i)} = G'^{(i)}. \tag{3.27}$$

Definition 3.19 (Stokes graded local system). A *Stokes graded local system* over (C, D, Θ) is a local system L over $\widehat{C}$ plus a Θ_ϕ-grading G_ϕ of L_ϕ for each $\phi \in \partial$ which is not a Stokes ray, obeying conditions:

- The grading G_ϕ is locally constant as ϕ varies.
- If ϕ is a Stokes ray, then the two gradings G, G' on the two sides of ϕ obey the Stokes condition.

We could unify these two conditions by requiring that for *any* ray ϕ the two gradings on the two sides of ϕ obey the Stokes condition, since when $\prec$ is trivial the Stokes condition says $G = G'$.

Example 3.20 (Stokes gradings on local models). A trivial example of a Stokes graded local system is obtained by considering our local model $(\mathcal{F}^\Theta, \nabla^{\Theta,m})$ in a neighborhood of a single boundary component ∂. Its local system of solutions comes by construction with Θ_ϕ-gradings, which are locally constant everywhere on ∂ (including at the Stokes rays).

[we could rephrase the notion of Stokes grading as an isomorphism to the local model which has jumps obeying a Stokes condition — this is the form we use later]

3.7. **Betti category with irregular singularities.** Now let $\widehat{S}$ be $\widehat{C}$ considered as a topological surface, and extract the "topological part" θ of Θ: we forget the map $q : \Theta \to \mathcal{E}$ and just remember the local system Θ, Stokes rays, multiplicities, and the partial orderings $\prec$ at each Stokes ray.

Definition 3.21 (Betti category with irregular singularities). Let $\mathcal{C}_{\mathrm{Betti}}(G, \widehat{S}, \theta)$ be the category of Stokes graded local systems over S valued in rank N $\mathbb{C}$-vector spaces, where for morphisms we take isomorphisms of local systems preserving the Stokes gradings.

Example 3.22 (Stokes graded local system for the Airy equation). Let L be the local system of solutions of the Airy equation. As we discussed in §1.2, there is a pair of distinguished solutions $\psi^{(i)}$ in each domain of C away from the Stokes rays. The label i corresponds to a local section of Θ. For any ϕ not a Stokes ray, the two $\psi^{(i)}$ span two distinct lines $G^{(i)}$ in L_ϕ, and thus give a Θ_ϕ-grading of L_ϕ. Moreover, at each Stokes ray, only the exponentially growing solution $\psi^{(i)}$ jumps, while the exponentially decaying $\psi^{(j)}$ varies continuously; this is the Stokes condition (3.27). Thus these gradings make L a Stokes graded local system. [should maybe move this example]

3.8. **Simple examples of moduli spaces.** As usual, we can consider the moduli space $\mathcal{M}_{\mathrm{Betti}}$ parameterizing objects of our category up to equivalence.

Example 3.23 (Configuration spaces of points on $\mathbb{CP}^1$). Suppose $C = \mathbb{CP}^1$, $D = \{\infty\}$, $N = 2$, and $\Theta_\infty = \{q^2 = z^s\}$. Then $\widehat{S}$ is the disc, with s Stokes rays on the boundary circle ∂, dividing it into s arcs. A Stokes graded local system over (C, D, Θ) is a rank 2 local system L on the disc, with a line decomposition on each boundary arc, labeled by the components of Θ_∞. To describe this object more explicitly, we can trivialize Θ_∞ away from some cuts where the sheets are exchanged. One convenient choice is to put one cut in the middle of each arc, and choose the trivializations so that $1 \prec 2$ at each Stokes ray.

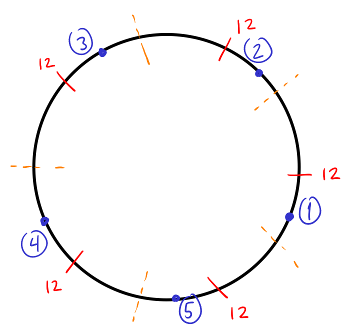

FIGURE 12. The disc $\widehat{S}$ in the case $s = 5$, with Stokes rays (red) and branch cuts (orange) marked. The grading of the fiber of L at the blue dot numbered r gives a decomposition $V = G_r^{(1)} \oplus G_r^{(2)}$.

Then we see that the Stokes graded local system is equivalent to a 2-dimensional vector space V plus s decompositions

$$V = G_r^{(1)} \oplus G_r^{(2)}, \qquad r = 1, \dots, s, \tag{3.28}$$

with the condition[22]

$$G_r^{(1)} = G_{r+1}^{(2)}. \tag{3.29}$$

More simply, it is equivalent to the s lines $G_r^{(1)} \subset V$, obeying the condition

$$G_r^{(1)} \neq G_{r+1}^{(1)}. \tag{3.30}$$

This gives an identification[23]

$$\mathcal{M}_{\text{Betti}}(G, \widehat{S}, \theta) \simeq \text{Conf}_s^{\text{cyc}}(\mathbb{CP}^1), \tag{3.31}$$

where by definition

$$\text{Conf}_s^{\text{cyc}}(\mathbb{CP}^1) = \{(z_1, \ldots, z_s) \in (\mathbb{CP}^1)^s : z_r \neq z_{r+1} \text{ for all } r\} / \text{PGL}(2, \mathbb{C}). \tag{3.32}$$

Exercise 3.8 (Configuration spaces of flags in $\mathbb{C}^N$). Suppose $C = \mathbb{CP}^1$, $D = \{\infty\}$, $\Theta_\infty = \{(N+1-2i)z^{\frac{s}{2}}\}_{i=1}^N$. Then $\widehat{S}$ is the disc again, divided into s arcs by s Stokes rays. Show that in this case $\mathcal{M}_{\text{Betti}}(G, C, D, \Theta)$ is the configuration space of s complete flags in $\mathbb{C}^N$, such that any two consecutive flags are in generic position. (When $N = 2$ this reduces to Example 3.23.) How do you recover the s gradings from the s flags?

Exercise 3.9 (Grassmannian quotients). Suppose $C = \mathbb{CP}^1$, $D = \{\infty\}$, $N = 3$, and $\Theta_\infty = \{q^3 = z^k\}$. Then $\widehat{S}$ is the disc again, with $s = 2k$ Stokes rays and divided into s arcs. Show that in this case $\mathcal{M}_{\text{Betti}}(G, C, D, \Theta)$ is a Grassmannian quotient:

$$\mathcal{M}_{\text{Betti}}(G, C, D, \Theta) \simeq \text{Gr}(3, k) / (\mathbb{C}^\times)^k \tag{3.33}$$

3.9. **Stokes matrices.** In the examples so far, we got some simplification from the fact that there was no monodromy around the boundary component ∂. More generally, this won't be the case. To organize things we introduce matrices which keep track of the difference between consecutive gradings:

Proposition 3.24 (Stokes matrix). Suppose V is a vector space and I a partially ordered multiset. Suppose G and G' are two I-gradings of V which obey the Stokes condition. Then there is a unique $S^{G \to G'} \in \text{Aut}(V)$ with the following properties:

- $S^{G \to G'}(G^{(i)}) = G'^{(i)}$.
- Decompose $S^{G \to G'}$ into blocks $S_{ji} : G^{(i)} \to G^{(j)}$. Then:
  - $S_{ii} = \mathbf{1}_{G^{(i)}}$,
  - $S_{ji} = 0$ unless $j \preceq i$.

(So S can map $G^{(i)} \to G^{(j)}$ nontrivially only if $j \preceq i$, and the interesting parts are the off-diagonal parts, where $j \prec i$. The suggestion of [12] is to think of $\prec$ as a little arrow; this reminds you which way the map goes.)

Conversely, given a matrix $S \in \text{Aut}(V)$ with a block decomposition as above, $S = S^{G \to G'}$ for some I-grading G' such that (G, G') obey the Stokes condition.

[22] We always consider r mod s, so this includes the condition $G_s^{(1)} = G_1^{(2)}$.

[23] We emphasize that the $\mathbb{CP}^1$ appearing here is not to be identified with C, but rather with the projective space of lines in V.

Proof. The Stokes condition says $G'^{(\preceq i)} = G^{(\preceq i)}$. Applying this to all $j \prec i$ and taking the direct sum, we also have $G'^{(\prec i)} = G^{(\prec i)}$. We define $S \in \mathrm{Aut}(V)$ so that $S|_{G^{(i)}}$ is the composite

$$G^{(i)} \to G^{(\preceq i)}/G^{(\prec i)} = G'^{(\preceq i)}/G'^{(\prec i)} \to G'^{(i)}. \tag{3.34}$$

Concretely, this composite maps $\psi \in G^{(i)}$ to the unique element $S(\psi) \in G'^{(i)}$ which is of the form

$$S(\psi) = \psi - p(\psi) \in G'^{(i)}, \qquad p(\psi) \in G^{(\prec i)}. \tag{3.35}$$

Then let $S^{G \to G'} = S$. The desired properties follow directly from (3.35).

For the converse define G' by $G'^{(i)} = S(G^{(i)})$; the block structure of S then implies that (G, G') obey the Stokes condition. □

We can describe the off-diagonal component $p : G^{(i)} \to G^{(\prec i)} = G'^{(\prec i)}$ which appeared in (3.35) as the projection map induced by the grading G'.

Example 3.25 (Stokes matrix in simplest case). Suppose $I = \{1, 2\}$ with the standard ordering $1 \prec 2$. Then an I-grading means a decomposition $G = G^{(1)} \oplus G^{(2)}$, and the Stokes condition says $G^{(1)} = G'^{(1)}$. The map $S^{G \to G'}$ given by Proposition 3.24 is of the form

$$S^{G \to G'} = \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} \tag{3.36}$$

relative to the grading G; the action of S and the projection $p : G^{(2)} \to G^{(1)}$ are indicated in Figure 13.

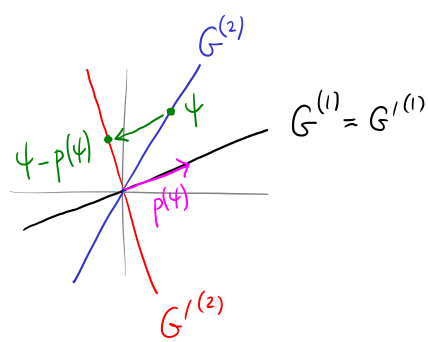

FIGURE 13. The off-diagonal component p of g.

Exercise 3.10. Suppose $I = \{1, 2\}$ with the standard ordering $1 \prec 2$, as in Example 3.25. Also suppose all multiplicities are 1, so $G^{(1)}$, $G^{(2)}$, $G'^{(2)}$ are all 1-dimensional. Let $\psi^{(1)} \in G^{(1)}$, $\psi^{(2)} \in G^{(2)}$, $\psi'^{(2)} \in G'^{(2)}$ be nonzero elements. Then prove that

$$S^{G \to G'}(\psi^{(2)}) = \frac{\psi^{(2)} \wedge \psi^{(1)}}{\psi'^{(2)} \wedge \psi^{(1)}}\, \psi'^{(2)}. \tag{3.37}$$

Exercise 3.11. Suppose $I = \{1, 2, 3\}$ with the partial ordering $1 \prec 2$. Also suppose all multiplicities are 1. Let $\psi^{(1)} \in G^{(1)}$, $\psi^{(2)} \in G^{(2)}$, $\psi'^{(2)} \in G'^{(2)}$, $\psi^{(3)} \in G^{(3)}$ be nonzero elements. Then prove that [check]

$$S^{G\to G'}(\psi^{(2)}) = \frac{\psi^{(2)} \wedge \psi^{(1)} \wedge \psi^{(3)}}{\psi'^{(2)} \wedge \psi^{(1)} \wedge \psi^{(3)}}\, \psi'^{(2)} \,. \tag{3.38}$$

Now we consider a Stokes graded local system L in the neighborhood of a boundary component ∂. Introduce a branch cut at some $\phi_0 \in \partial$ and let $V = L(\partial \setminus \phi_0)$, $I = \Theta(\partial \setminus \phi_0)$. Suppose there are s Stokes rays on ∂. Pick an orientation on ∂; then going around ∂ starting from ϕ_0 we get $s+1$ I-gradings $G_1, \dots, G_{s+1}$ of V. For $1 \le r \le s$, (G_r, G_{r+1}) obey a Stokes condition associated to the ordering $\prec_r$ attached to the r-th Stokes ray. Let $M : V \to V$ be the monodromy automorphism (in the direction given by the orientation), and let $S_r = S^{G_r \to G_{r+1}}$. Let $G = G_1$. Let $\sigma \in \mathrm{Aut}(\Theta_{\phi_0})$ be the monodromy permutation. Then we also have the permuted grading G^σ defined by $(G^\sigma)^{(i)} = G^{\sigma(i)}$; it obeys $G^\sigma = MG_{s+1}$.

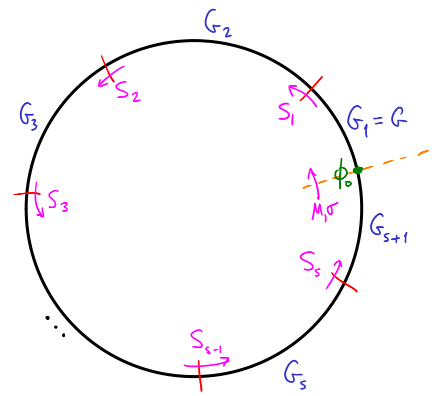

FIGURE 14. A boundary component ∂, with s Stokes rays, s Stokes matrices, $s+1$ gradings of $V = L(\partial \setminus \phi_0)$, and monodromies M, σ for the local systems L, Θ.

So, our data can be summarized as:

- A vector space V,
- An I-grading G of V,
- Automorphisms $S_1, \dots, S_s, M \in \mathrm{Aut}(V)$,

obeying two kinds of constraints:

- Each S_r obeys a triangularity condition relative to the grading G_r, dictated by the ordering $\prec_r$, as in Proposition 3.24.
- We consider the composite ("formal monodromy")

$$\Lambda = MS_s \cdots S_2 S_1 \in \mathrm{Aut}(V). \tag{3.39}$$

  Then Λ maps the I-grading G to the permuted grading G^σ.

For explicit formulas, it is convenient to write everything with respect to the fixed grading G. To do this we exchange the S_r for the matrices

$$U_r = S^{G \to (S_{r-1}\cdots S_1)^{-1} G_{r+1}} = (S_{r-1} \cdots S_1)^{-1} S_r (S_{r-1} \cdots S_1). \tag{3.40}$$

Then each U_r obeys the $\prec_r$-triangularity condition relative to G, and the constraint becomes

$$\Lambda = MU_1 \cdots U_s. \tag{3.41}$$

Example 3.26 (Stokes matrix description of the Weber equation). We reconsider Example 3.23 in the case $s = 4$. In this case the Stokes matrices alternate between upper and lower triangular, the permutation σ is the identity, and the total monodromy $M = \mathbf{1}$. Thus the constraint is

$$\Lambda = U_1 U_2 U_3 U_4 \tag{3.42}$$

which we can write more explicitly relative to G as

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \tag{3.43}$$

where $\lambda \in \mathbb{C}^\times$, $a, c : G^{(2)} \to G^{(1)}$, $b, d : G^{(1)} \to G^{(2)}$. This gives the conditions

$$a + \lambda c = 0, \quad \lambda b + d = 0, \quad 1 + bc = \lambda^{-1}. \tag{3.44}$$

The space $\mathcal{M}_{\mathrm{Betti}}(G, \widehat{S}, \theta)$ is parameterized by matrices U_r of this form up to equivalence: this means overall conjugation by a diagonal matrix. As long as $\lambda \neq 1$, the only invariant up to equivalence is the eigenvalue of formal monodromy, $\lambda \in \mathbb{C}^\times$. When $\lambda = 1$ we have three possibilities up to equivalence: either ($a = 0$ and $b \neq 0$), ($a \neq 0$ and $b = 0$), or ($a = b = 0$). Thus we can picture $\mathcal{M}_{\mathrm{Betti}}$ as a non-Hausdorff space, $\mathbb{C}^\times$ with the point $\lambda = 1$ tripled.

Exercise 3.12. In Example 3.23 we showed that $\mathcal{M}_{\mathrm{Betti}}$ in this case is $\mathrm{Conf}_4^{\mathrm{cyc}}(\mathbb{CP}^1)$. Show that λ is a cyclically ordered cross-ratio, $\lambda = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$. What is the meaning of the tripled point at $\lambda = 1$?

Example 3.27 (Stokes matrix description of the Airy equation). We reconsider Example 3.23 in the case $s = 3$. Again the Stokes matrices alternate between upper and lower triangular, and again the total monodromy $M = \mathbf{1}$. The monodromy σ of the local system Θ_∞ is the nontrivial permutation, so in this case Λ will be an off-diagonal matrix instead of diagonal. The constraint is

$$\Lambda = U_1 U_2 U_3, \tag{3.45}$$

i.e.

$$\begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \tag{3.46}$$

for some $\lambda : G^{(2)} \to G^{(1)}$. This determines the Stokes matrices as

$$a = c = \lambda, \quad b = -\lambda^{-1}. \tag{3.47}$$

Moreover, all values of λ are equivalent. This agrees with the fact that $\mathcal{M}_{\mathrm{Betti}}$ in this case is $\mathrm{Conf}_3^{\mathrm{cyc}}(\mathbb{CP}^1)$, which is a point.

Example 3.28 (One regular and one irregular singularity). Now we consider a new example. Suppose $C = \mathbb{CP}^1$, $D = \{0, \infty\}$, $\Theta_0 = \{0, 0, \dots, 0\}$, $\Theta_\infty = \{\frac{u_1}{z}, \dots, \frac{u_N}{z}\}$, with all $u_i \in \mathbb{R}_+$ and $u_1 > u_2 > \cdots > u_n$. In this case the circle ∂_∞ has two Stokes rays, one with the ordering $1 \prec 2 \prec \cdots \prec N$ and one with $N \prec \cdots \prec 2 \prec 1$.

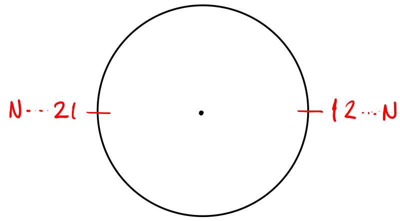

FIGURE 15. The disc with one regular and one irregular singularity, and Θ as given in the main text.

Thus our constraint becomes

$$\Lambda = MU_1U_2 \tag{3.48}$$

where U_1 is lower-unipotent and U_2 upper-unipotent. We can rewrite this to say M is determined by the other data:

$$M = \Lambda U_2^{-1}U_1^{-1} \tag{3.49}$$

Altogether then

$$\mathcal{M}_{\text{Betti}}(G, \widehat{S}, \theta) = (T \times U_- \times U_+)/T \tag{3.50}$$

with the T-action given by simultaneous conjugation. Equivalently we could say that $\mathcal{M}_{\text{Betti}}(G, \widehat{S}, \theta)$ consists of those matrices M which admit such a factorization — also called the "big cell" in the Bruhat decomposition of $G = \mathrm{GL}(N, \mathbb{C})$ — modulo conjugation by T. Note that this is different from what we would get with *regular* singularities at 0 and ∞: in that case we would get arbitrary matrices M modulo conjugation by the whole group G.

3.10. **Borel summation.** Our basic tool for constructing Stokes gradings will be Borel summation. Although our ultimate interest is in constructing sections of vector bundles, we begin with the case of a function.

Definition 3.29 (Borel transform). Suppose given a formal series

$$f(t) = \sum_{n=0}^{\infty} c_n t^n . \tag{3.51}$$

The *Borel transform* of this series is

$$\mathcal{B}f(\zeta) = \sum_{n=0}^{\infty} \frac{c_n}{n!}\zeta^n . \tag{3.52}$$

Definition 3.30 (Borel summation in direction ϑ). Suppose that $f(t)$ is a formal series, such that $\mathcal{B}f(\zeta)$ converges in some disc around $\zeta = 0$ and admits analytic continuation in an open sector around the ray $\mathrm{e}^{\mathrm{i}\vartheta}[0, \infty)$, obeying a bound of the form $|\mathcal{B}f(\zeta)| \ll \mathrm{e}^{M|\zeta|}$ for some $M > 0$. Then we define

$$\mathcal{S}^{\vartheta} f(t) = t^{-1} \int_0^{\mathrm{e}^{\mathrm{i}\vartheta}\infty} \mathrm{e}^{-\zeta/t}\, \mathcal{B}f(\zeta)\, \mathrm{d}\zeta \tag{3.53}$$

for any t with $|t| < M^{-1}\cos(\vartheta - \arg t)$.

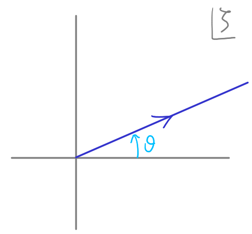

FIGURE 16. The contour of integration in the ζ-plane for Borel summation.

Proposition 3.31 (Holomorphy of the Borel sum). With f as in Definition 3.30, $\mathcal{S}^\vartheta f(t)$ is a holomorphic function in the domain $\{t : |t| < M^{-1}\cos(\vartheta - \arg t)\}$.

Proof. Differentiate under the integral sign, using the bound on $|\mathcal{B}f(\zeta)|$ to get control. $\square$

In particular, if $|\mathcal{B}f(\zeta)| \ll \mathrm{e}^{M|\zeta|}$ for $\zeta \in \mathrm{e}^{\mathrm{i}\vartheta}[0,\infty)$ and every $M > 0$, then $\mathcal{S}^\vartheta f(t)$ exists in the whole half-plane $\mathbb{S}_{(\vartheta-\frac{\pi}{2},\vartheta+\frac{\pi}{2})}$ and is holomorphic there.

To get beyond a half-plane we should optimize our choice of ϑ, as follows:

Definition 3.32 (Borel summation). We define

$$\mathcal{S}f(t) = \mathcal{S}^{\vartheta=\arg t} f(t) \tag{3.54}$$

for any t where this is defined.

Another convenient representation is

$$\mathcal{S}f(t) = |t|^{-1} \int_0^\infty \mathrm{e}^{-u/|t|} \mathcal{B}f(\mathrm{e}^{\mathrm{i}\arg t} u)\,\mathrm{d}u. \tag{3.55}$$

Here are some basic facts about Borel summation.

Proposition 3.33 (Borel summation of convergent series). If the series $f(t)$ converges for $|t| < R$, then $\mathcal{S}f(t) = f(t)$ for $|t| < R$.

Proof. First note the integral identity[24]

$$\int_0^\infty \mathrm{e}^{-\zeta} \zeta^n \,\mathrm{d}\zeta = n! \tag{3.56}$$

which after a change of variables $\zeta \to \zeta/t$ and dividing by $n!$ gives

$$t^{-1} \int_0^{t\infty} \mathrm{e}^{-\zeta/t} \frac{\zeta^n}{n!} \,\mathrm{d}\zeta = t^n. \tag{3.57}$$

Now suppose

$$f(t) = \sum_{n=0}^\infty c_n t^n \tag{3.58}$$

convergent for $|t| < R$. In this case

$$\mathcal{B}f(\zeta) = \sum_{n=0}^\infty \frac{c_n}{n!} \zeta^n \tag{3.59}$$

[24] This identity can be proven by repeated integration by parts, a special case of the proof of the functional equation of $\Gamma(z)$.

is entire and bounded by $Ce^{|\zeta|/R}$ for some C. Thus for $|t| < R$ we can write

$$\mathcal{S}f(t) = t^{-1} \int_0^{t\infty} e^{-\zeta/t} \mathcal{B}f(\zeta)\, d\zeta = t^{-1} \int_0^{t\infty} e^{-\zeta/t} \sum_{n=0}^{\infty} \frac{c_n}{n!} \zeta^n\, d\zeta = \sum_{n=0}^{\infty} c_n t^n = f(t) \tag{3.60}$$

as desired. □

Proposition 3.34 (Borel summation is a map of differential algebras). If $\mathcal{S}f(t)$ and $\mathcal{S}g(t)$ exist on a domain U, then:

- $\mathcal{S}(f+g) = \mathcal{S}(f) + \mathcal{S}(g)$.
- $\mathcal{S}(\partial_t f) = \partial_t \mathcal{S}(f)$.
- $\mathcal{S}(fg) = \mathcal{S}(f)\mathcal{S}(g)$.

Exercise 3.13. Prove Proposition 3.34.

Definition 3.35 (Sector). A *sector* (around $z = 0$) is a subset

$$\mathbb{S}_{I,R} = \{z : \arg(z) \in I, |z| < R\} \subset \mathbb{C}, \tag{3.61}$$

for some interval $I \subset \mathbb{R}$, and $R \in \mathbb{R}_+ \cup \{\infty\}$. We also write $\mathbb{S}_I$ for $\mathbb{S}_{I,\infty}$. The *opening angle* of $\mathbb{S}_{I,R}$ is the length of I.

Definition 3.36 (Asymptotic series). Given a function ψ defined on some open sector $U \subset \mathbb{C}$, and a sequence (c_n) in $\mathbb{C}$, we say that

$$\psi(t) \sim \sum_{n=0}^{\infty} c_n t^n \qquad \text{as } t \to 0 \text{ in } U \tag{3.62}$$

just if, for all $N \geq 0$,

$$\lim_{t\to 0} t^{-N} \left(\psi(t) - \sum_{n=0}^{N} c_n t^n \right) = 0. \tag{3.63}$$

Note that an asymptotic series need not be a convergent series, and it does not determine the function ψ uniquely. However, there is a slightly stronger notion which does have some uniqueness properties:

Definition 3.37 (Gevrey-1 asymptotics). An asymptotic expansion (3.62) is called *Gevrey*-1 if for every closed subsector $S \subset U$ there exist $C, K > 0$ such that in S

$$\left| \psi(t) - \sum_{n=0}^{N} c_n t^n \right| \leq CK^{N+1}(N+1)!|t|^{N+1} \tag{3.64}$$

Proposition 3.38 (Uniqueness of functions with Gevrey-1 asymptotics on big sectors). Suppose U is a sector with opening angle $> \pi$, and $\psi_i \sim \psi_{\text{formal}}$ is a Gevrey-1 asymptotic expansion on U for $i = 1, 2$. Then $\psi_1 = \psi_2$.

Proof. [...] □

Proposition 3.39 (Borel summation has the expected asymptotic series). If $f(t)$ is a formal series in t, such that $\mathcal{S}f(t)$ exists on a domain U, then $\mathcal{S}f(t) \sim f(t)$ as $t \to 0$ in U, and this expansion is Gevrey-1.

(Note that this would be immediate if we were allowed to replace $\mathcal{B}f(\zeta)$ by the formal series and then integrate term by term in the integral defining $\mathcal{S}f(t)$. Informally speaking, what we are saying is that even though this term-by-term integration does not produce a convergent series, it does produce a correct asymptotic series.)

Proof. Suppose

$$f(t) = \sum_{n=0}^{\infty} c_n t^n \,. \tag{3.65}$$

Using Proposition 3.33 applied to the first N terms of the series we see that

$$\mathcal{S}f(t) - \sum_{n=0}^{N} c_n t^n = |t|^{-1} \int_0^\infty \mathrm{e}^{-u/|t|} \Phi_N(\mathrm{e}^{\mathrm{i}\arg t} u)\, \mathrm{d}u \tag{3.66}$$

where

$$\Phi_N(\zeta) = \mathcal{B}f(\zeta) - \sum_{n=0}^{N} \frac{c_n}{n!} \zeta^n \,. \tag{3.67}$$

What remains is to show that the RHS of (3.66) decays faster than t^N as $t \to 0$.

For this we fix $\epsilon > 0$ and divide up the integral as $I = I_\epsilon^{\mathrm{near}} + I_\epsilon^{\mathrm{far}}$. First consider

$$I_\epsilon^{\mathrm{far}} = |t|^{-1} \int_\epsilon^\infty \mathrm{e}^{-u/|t|} \Phi_N(\mathrm{e}^{\mathrm{i}\arg t} u)\, \mathrm{d}u \,. \tag{3.68}$$

Because $|\Phi_N(\zeta)| < \mathrm{e}^{M|\zeta|}$ we can estimate this for $|t| < 1/M$ as

$$|I_\epsilon^{\mathrm{far}}| < |t|^{-1} \int_\epsilon^\infty \mathrm{e}^{-u(1/|t|-M)}\, \mathrm{d}u = \frac{\mathrm{e}^{M\epsilon}}{1 - M|t|} \mathrm{e}^{-\epsilon/|t|} \ll |t|^N \text{ as } t \to 0 \quad . \tag{3.69}$$

Now suppose ϵ was chosen smaller than the radius of convergence of the series defining $\mathcal{B}f(\zeta)$. Then, for $|\zeta| < \epsilon$, we have $|\Phi_N(\zeta)| < C|\zeta|^{N+1}$ for some C, which moreover can be taken independent of N. Then we have

$$|I_\epsilon^{\mathrm{near}}| < C|t|^{-1} \int_0^\epsilon \mathrm{e}^{-u/|t|} u^{N+1}\, \mathrm{d}u < C|t|^{-1} \int_0^\infty \mathrm{e}^{-u/|t|} u^{N+1}\, \mathrm{d}u = C(N+1)!|t|^{N+1} \ll |t|^N \,. \tag{3.70}$$

Combining (3.69) and (3.70) gives the desired result. □

In practice the integral defining the Borel sum $\mathcal{S}f(t)$ may exist even for some t which are outside the radius of convergence of the original series $f(t)$. For instance:[25]

Exercise 3.14. Consider the geometric series $f(t) = \sum_{n=0}^\infty t^n$. Prove directly that the integral defining $\mathcal{S}f(t)$ converges for $\mathrm{Re}(t) < 1$ and that it agrees with the function $\frac{1}{1-t}$ on that domain.

Thus Borel summation goes beyond what ordinary summation can do, for series with a finite radius of convergence. The importance of Borel summation for us lies mainly in the fact that this continues to be true even when the radius of convergence is zero. The next example illustrates this:

[25] A caution: mere existence of the integral does not *a priori* imply that it is analytic outside the domain guaranteed in Proposition 3.31. I am not sure how big a problem this is in practice.

Example 3.40 (Borel summation for a factorial series). We consider the formal series

$$f(t) = \sum_{n=0}^{\infty} n! t^n \,. \tag{3.71}$$

This series has zero radius of convergence. Its Borel transform is

$$\mathcal{B}f(\zeta) = \sum_{n=0}^{\infty} \zeta^n \,, \tag{3.72}$$

which converges in the unit disc to

$$\mathcal{B}f(\zeta) = \frac{1}{1-\zeta} \,. \tag{3.73}$$

Crucially, this function has analytic continuation along every ray except the ray $\mathbb{R}_+$, with subexponential growth. Thus we may define the Borel summation by

$$\mathcal{S}f(t) = t^{-1} \int_0^{e^{i \arg(t)} \infty} e^{-\zeta/t} \frac{1}{1-\zeta} \, d\zeta \,. \tag{3.74}$$

Call this function $\phi(t)$. It is defined in the slit plane $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$.

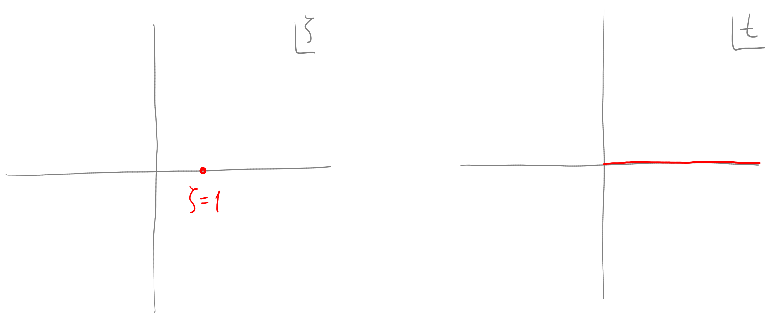

FIGURE 17. Left: the ζ-plane, with the pole of $\mathcal{B}f(\zeta)$ marked. Right: the t-plane, with the slit marked; $\mathcal{S}f(t)$ is defined in the complement of the slit.

Moreover, $\phi(t)$ is holomorphic in its domain. Indeed, the variable t appears in two places: both in the integrand and in the definition of the contour. The t dependence in the integrand is analytic for the usual reason — the integrand is holomorphic in t and the integral is sufficiently well convergent to exchange integration and differentiation. The t dependence in the definition of the contour is locally constant, because the integrand dies off fast enough at $\zeta = \infty$ to permit contour deformation there.

Now, $\phi(t)$ is not defined for $t \geq 0$, because in that case there is a singularity on the integration contour. What happens when we cross this locus? Letting

$$\phi^{\pm}(t) = \lim_{\epsilon \to 0^+} \phi(t \pm i\epsilon), \tag{3.75}$$

we have

$$\Delta = \phi^+(t) - \phi^-(t) = -t^{-1} \oint_1 e^{-\zeta/t} \frac{1}{1-\zeta} \, d\zeta = 2\pi i \, t^{-1} e^{-1/t}. \tag{3.76}$$

Some remarks:

- To summarize: there is a perfectly analytic function $\phi(t)$ on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$, whose asymptotic series around $t = 0$ is the divergent series $\sum_{n=0}^{\infty} n! t^n$. At the positive real axis, $\phi(t)$ jumps by $\Delta(t) = 2\pi\mathrm{i}\, t^{-1} \mathrm{e}^{-1/t}$. This jump is imperceptible in the $t \to 0$ asymptotic expansion near the real axis, as it must be since that expansion doesn't change when we cross the axis.
- We emphasize that the real axis is not a natural boundary for ϕ. Indeed, the fact that $\Delta(t)$ is analytic in $\mathbb{C}^\times$ implies that one could analytically continue $\phi(t)$ across the real axis, and indeed along an arbitrary path in $\mathbb{C}^\times$. On a general sheet this continuation gives $\phi_n(t) = \phi(t) + n\Delta(t)$ for some $n \in \mathbb{Z}$. But this analytically continued function does *not* have the desired asymptotic expansion as $t \to 0$: the extra shift $\Delta(t)$ is exponentially large in the left half-plane. In short, what we are seeing here is not a failure of analytic continuation, rather a conflict between analytic continuation and the asymptotic expansion. This conflict is a manifestation of the Stokes phenomenon.
- According to Proposition 3.39 the asymptotics $\phi(t) \sim f(t)$ are Gevrey-1 in the slit plane, which is a sector of radius $> \pi$. Using Proposition 3.38, it follows that $\phi(t)$ is the unique function with this property. Thus the Stokes phenomenon experienced by $\phi(t)$ is really something intrinsic to the series $f(t)$, rather than an artifact of the procedure of Borel summation.
- The series $f(t)$ is a formal solution of the inhomogeneous differential equation
$$t\partial_t(t\psi(t)) - \psi(t) = -1 \tag{3.77}$$
and $\phi(t)$ is an actual solution.[26] The shift $\Delta(t)$ solves the homogeneous equation.

Exercise 3.15. Repeat this analysis for the series $f(t) = \sum_{n\geq 0} n! a^{-n} t^n$, for some $a \in \mathbb{C}$. Show that in this case the pole of $\mathcal{B}f(\zeta)$ appears at $\zeta = a$, and the corresponding Stokes jump scales like $\mathrm{e}^{-a/t}$.

Example 3.41 (Borel summation for the gamma function). Recall the Bernoulli numbers $B_m \in \mathbb{Q}$ defined by
$$\frac{t}{\mathrm{e}^t - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} t^m. \tag{3.78}$$
Define for $t \notin \mathbb{R}_-$
$$\xi(t) = \log\Gamma(t^{-1}) - \left(-t^{-1}\log t - t^{-1} + \frac{1}{2}\log(2\pi t)\right) \tag{3.79}$$
Then $t^{-1}\xi(t)$ admits an asymptotic series[27]
$$t^{-1}\xi(t) \sim f(t) = \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} t^{2n-2} \qquad \text{as } t \to 0 \text{ in } \mathbb{R}_+ \tag{3.80}$$

[26] The fact that this is an inhomogeneous equation gets around our usual dogma that Stokes phenomena only occur in rank $N > 1$ — that dogma applies only to homogeneous equations.

[27] I encourage you to look at Section 5.4 in [15] for a marvelous discussion of the virtues of this asymptotic series.

The asymptotics $|B_{2n}| \sim 4\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}$ imply this series has zero radius of convergence. Now we try Borel summation: we have

$$\mathcal{B}f(\zeta) = \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\zeta^{2n-2} \tag{3.81}$$

which converges in a disc around $\zeta = 0$ and, using (3.78), meromorphically continues to

$$\mathcal{B}f(\zeta) = \frac{1}{2\zeta^2}\left(\frac{\zeta}{e^{\zeta}-1} - \frac{\zeta}{e^{-\zeta}-1} - 2\right) = \frac{1}{2\zeta}\coth\left(\frac{\zeta}{2}\right) - \frac{1}{\zeta^2}. \tag{3.82}$$

This function has poles only at $\zeta = 2\pi i n$, $n \neq 0$. It follows that $\mathcal{S}f(t)$ exists for $t \notin i\mathbb{R}$. Explicitly

$$\mathcal{S}f(t) = t^{-1}\int_0^{t\infty} e^{-\zeta/t}\left(\frac{1}{2\zeta}\coth\left(\frac{\zeta}{2}\right) - \frac{1}{\zeta^2}\right) d\zeta. \tag{3.83}$$

Moreover, for $\operatorname{Re} t > 0$ one finds $\mathcal{S}f(t) = t^{-1}\xi(t)$ on the nose; this is Binet's first formula for $\log\Gamma$. (If one has an independent proof of Binet's first formula, as discussed e.g. in [16], then this gives a derivation of the asymptotic series (3.80).)

For $t \in \mathbb{R}_-$, on the other hand, $\xi(t)$ does not have the asymptotic series (3.80) — indeed it is not well defined at all thanks to the poles of $\Gamma(t^{-1})$. Instead we note that $\mathcal{S}f(t) = \mathcal{S}f(-t)$ (this goes back to the fact that the series $f(t) = f(-t)$) and thus

$$\mathcal{S}f(t) = \begin{cases} t^{-1}\xi(t) & \text{for } \operatorname{Re}(t) > 0, \\ -t^{-1}\xi(-t) & \text{for } \operatorname{Re}(t) < 0. \end{cases} \tag{3.84}$$

The difference between the two, for $t \in i\mathbb{R}_+$, can be computed using the reflection formula

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)}, \tag{3.85}$$

which gives

$$\Delta(t) = -t^{-1}\xi(-t) - t^{-1}\xi(t) = [\ldots] \tag{3.86}$$

Note that this function is very well behaved on the imaginary axis.

Exercise 3.16. Compute $\Delta(t)$ directly using the integral formula (3.83), by summing over residues; check that the result matches (3.86). (This gives an alternative proof of the reflection formula.)

In preparation for the next example we note that the Borel transform is well behaved with respect to differential operators:

Exercise 3.17. Prove that $\mathcal{B}(t\partial_t f(t)) = \zeta\partial_\zeta(\mathcal{B}f(t))$, and $\mathcal{B}(\partial_t f(t)) = \partial_\zeta(\zeta\partial_\zeta\mathcal{B}(f(t)))$. (In other words, the Borel transform takes $t\partial_t \mapsto \zeta\partial_\zeta$, and $\partial_t \mapsto \partial_\zeta(\zeta\partial_\zeta)$.)

Exercise 3.18. Prove that, if $f(t)$ has zero constant term, then $\mathcal{B}(t^{-1}f(t)) = \partial_\zeta\mathcal{B}(f(t))$. (In other words, with this caveat, the Borel transform takes $t^{-1} \mapsto \partial_\zeta$.)

Exercise 3.19. Suppose that $f(t)$ is annihilated by a differential operator $D \in \mathbb{C}[t\partial_t, t^{-1}]$. Prove that $\mathcal{B}f(\zeta)$ is annihilated by the operator $\mathcal{F}(D)$, where $\mathcal{F}$ takes $t^{-1} \mapsto \partial_\zeta$, $t\partial_t \mapsto \zeta\partial_\zeta$. [unchecked]

One way of understanding these properties is the following formal analogy. Recall the *inverse Laplace transform*

$$(\mathcal{I}\psi)(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_{s-\mathrm{i}\infty}^{s+\mathrm{i}\infty} \mathrm{e}^{x\zeta} x^{-1} \psi(x^{-1}) \mathrm{d}x \tag{3.87}$$

where $s \in \mathbb{R}$ is chosen large enough that the integrand is holomorphic for $\mathrm{Re}(x) \geq s$. (Thus $\mathcal{I}\psi$ is only defined when $\psi(x^{-1})$ is holomorphic for large enough $\mathrm{Re}(x)$.) [should also describe this after the change of variables $t = x^{-1}$]

Exercise 3.20. Show that if $\psi(t) = t^n$, then $\mathcal{I}\psi(\zeta) = \frac{\zeta^n}{n!}$ when $\zeta \in \mathbb{R}_+$.

Thus the Borel transform can be thought of as a formal version of the inverse Laplace transform. From this point of view, the properties in Exercise 3.17 are consequences of differentiation under the integral sign and integration by parts.

This point of view also suggests a slight extension, which will be convenient for the next example. We will want to consider expressions of the form[28]

$$f(t) = \mathrm{e}^{a/t} \sum_{n=0}^{\infty} c_n t^{n+\alpha} . \tag{3.88}$$

Exercise 3.21. Show that if $\psi(t) = \mathrm{e}^{a/t} t^{\alpha}$, then $\mathcal{I}\psi(\zeta) = \frac{(\zeta+a)^{\alpha}}{\Gamma(\alpha+1)}$ when $\zeta + a \in \mathbb{R}_+$ (where we choose the principal branch).

With this motivation, we define the Borel transform of such an expression to be

$$\mathcal{B}f(\zeta) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n+\alpha+1)} (\zeta + a)^{n+\alpha}. \tag{3.89}$$

Exercise 3.22. Prove that the properties in Exercise 3.17, Exercise 3.18, Exercise 3.19 continue to hold for this extended version of the Borel transform.

For such a transseries the Borel summation procedure is slightly modified: the lower limit of integration should be $-a$ instead of zero, i.e.

$$\mathcal{S}^{\vartheta} f(t) = t^{-1} \int_{-a}^{-a+\infty \mathrm{e}^{\mathrm{i}\vartheta}} \mathrm{e}^{-\zeta/t} \mathcal{B}f(\zeta) \, \mathrm{d}\zeta. \tag{3.90}$$

Example 3.42 (Borel summation for the Airy equation). Now we consider the transseries

$$\psi^{(1)}_{\mathrm{formal}} = \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \mathrm{e}^{-\frac{2}{3} z^{\frac{3}{2}}} \sum_{n=0}^{\infty} c_n (-1)^n z^{-\frac{3}{2}n}, \tag{3.91}$$

$$\psi^{(2)}_{\mathrm{formal}} = \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \mathrm{e}^{\frac{2}{3} z^{\frac{3}{2}}} \sum_{n=0}^{\infty} c_n z^{-\frac{3}{2}n}, \tag{3.92}$$

$$c_n = \left(\frac{3}{4}\right)^n \frac{\Gamma(n+\frac{5}{6})\Gamma(n+\frac{1}{6})}{n!} . \tag{3.93}$$

[28] These are examples of *transseries*, although we will not develop this language formally; see e.g. [17] for an introduction.

According to Exercise 1.5 these are formal solutions of the Airy equation (1.12). We want to discuss Borel summation of these series. To get them into our standard form we make a change of variables

$$t = z^{-\frac{3}{2}} \tag{3.94}$$

and remove the prefactor $\frac{1}{2\sqrt{\pi}}$, as well as a factor $t^{1/6}$, thus defining

$$\psi^{(1)}_{\mathrm{formal}}(t) = \mathrm{e}^{-\frac{2}{3}t^{-1}} \sum_{n=0}^{\infty} c_n (-1)^n t^n \tag{3.95}$$

This transseries has zero radius of convergence. Its Borel transform

$$\mathcal{B}\psi^{(1)}_{\mathrm{formal}}(\zeta) = \sum_{n=0}^{\infty} \frac{c_n(-1)^n}{n!} \left(\zeta - \frac{2}{3}\right)^n \tag{3.96}$$

is better behaved: the factorials cancel and so we are left with radius of convergence $R = \frac{4}{3}$, and similarly for $\mathcal{B}\psi^{(2)}$.

By itself, this is not good enough to guarantee that $\mathcal{S}\psi^{(1)}_{\mathrm{formal}}$ exists: for that we need to know that $\mathcal{B}\psi^{(1)}_{\mathrm{formal}}(\zeta)$ admits analytic continuation as $\zeta \to \infty$ along a generic ray from $\zeta = \frac{2}{3}$, and grows sufficiently slowly as $\zeta \to \infty$. Similarly for $\psi^{(2)}_{\mathrm{formal}}$, except that there we consider rays from $\zeta = -\frac{2}{3}$.

One way to get the necessary properties uses the fact that the $\mathcal{B}\psi^{(i)}_{\mathrm{formal}}(\zeta)$ both obey a differential equation. To see this, first rewrite the Airy equation (1.12) as

$$\left(z^2\partial_z^2 - z^3\right)\psi(z) = 0, \tag{3.97}$$

i.e.

$$\left((z\partial_z)^2 - z\partial_z - z^3\right)\psi(z) = 0. \tag{3.98}$$

In terms of the variable t we have $z\partial_z = -\frac{3}{2}t\partial_t$ and thus this equation becomes

$$\left(\frac{9}{4}(t\partial_t)^2 + \frac{3}{2}t\partial_t - t^{-2}\right)\psi(t) = 0. \tag{3.99}$$

Taking $\psi \to t^{-1/6}\psi$ shifts $t\partial_t \to t\partial_t + \frac{1}{6}$, after which the equation becomes

$$\left(\frac{9}{4}(t\partial_t)^2 + \frac{9}{4}t\partial_t + \frac{5}{16} - t^{-2}\right)\psi(t) = 0. \tag{3.100}$$

Using Exercise 3.22 this implies

$$\left(\frac{9}{4}(\zeta\partial_\zeta)^2 + \frac{9}{4}\zeta\partial_\zeta + \frac{5}{16} - \partial_\zeta^2\right)\mathcal{B}\psi(\zeta) = 0, \tag{3.101}$$

which we can rearrange to

$$\left(\partial_\zeta^2 + \frac{18\zeta}{9\zeta^2 - 4}\partial_\zeta + \frac{5}{36\zeta^2 - 16}\right)\mathcal{B}\psi(\zeta) = 0. \tag{3.102}$$

This equation has regular singularities at $\zeta = \pm\frac{2}{3}$ and at $\zeta = \infty$. Thus $\mathcal{B}\psi^{(1)}_{\mathrm{formal}}$ will indeed have analytic continuation along a generic ray to $\zeta = \infty$. From now on we let

$\mathcal{B}\psi^{(1)}_{\text{formal}}$ denote the continued function; it is analytic in the complement of a slit at $\zeta < -\frac{2}{3}$. Likewise we continue $\mathcal{B}\psi^{(2)}_{\text{formal}}$ to a function analytic in the complement of $\zeta > \frac{2}{3}$. [fig]

At each singularity the characteristic exponents (eigenvalues of the residue of the connection) are $(0,0)$. At $\zeta = \frac{2}{3}$, we see from the series representation (3.96) that $\mathcal{B}\psi^{(1)}_{\text{formal}}$ is an eigenvector; at $\zeta = -\frac{2}{3}$, similarly $\mathcal{B}\psi^{(2)}_{\text{formal}}$ is an eigenvector. Now what about the monodromy of $\mathcal{B}\psi^{(1)}_{\text{formal}}$ around $\zeta = -\frac{2}{3}$? Since the eigenvalues of the residue are equal, the generalized eigenvalues of the monodromy M around this point must also be equal: in the basis $(\mathcal{B}\psi^{(1)}_{\text{formal}}, \mathcal{B}\psi^{(2)}_{\text{formal}})$ we thus have

$$M = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \tag{3.103}$$

for some a. Concretely, this means that the jump of $\mathcal{B}\psi^{(1)}_{\text{formal}}$ across the cut emanating from $\zeta = -\frac{2}{3}$ is by $a\mathcal{B}\psi^{(2)}_{\text{formal}}$. It follows that the Borel sum $\mathcal{S}\psi^{(1)}_{\text{formal}}$ jumps by $\Delta = a\mathcal{S}\psi^{(2)}_{\text{formal}}$ when t crosses $\mathbb{R}_-$. Similarly $\mathcal{S}\psi^{(2)}_{\text{formal}}$ jumps by $\Delta = a'\mathcal{S}\psi^{(1)}_{\text{formal}}$ when t crosses $\mathbb{R}_+$. When translated back into the z-plane, these are the expected Stokes phenomena for solutions of the Airy equation.

[comment on computation of a, a'? by comparing with the intro seem to see they are $\pm \mathrm{i}$; how to calculate this directly from the Borel transformed series?]

Exercise 3.23. Do the analogous analysis of Borel summability for the higher-order Airy-like equation $(\partial_z^N + z)\psi = 0$. Show in particular that the Stokes jumps of the solutions are of the expected form: at a Stokes line with $i \prec j$, the solution $\psi^{(j)}$ jumps by a multiple of $\psi^{(i)}$. (The key point is to control the Jordan block structure of the monodromy of $\mathcal{B}\psi$ around the singularities in the ζ-plane; for this see Exercise 2.4.)

Exercise 3.24. Try doing the analogous analysis of Borel summability for the equation with higher-degree homogeneous potential, $(\partial_z^2 + z^k)\psi = 0$.

Exercise 3.25. Show that the Stokes graded local system corresponding to the equation in Exercise 3.24 is invariant under the action of rotating the boundary circle by 2 units. So this is an exceptional case where we can (almost) determine the Stokes graded local system, by pure symmetry considerations. The case $k = 3$ is already interesting. How many fixed points of this action are there in $\mathcal{M}_{\text{Betti}}$ in this case? [remark on convexity?]

Example 3.43 (Borel summation for a Schrödinger equation with polynomial potential). We consider a Schrödinger equation with $P(z) = z^3 + az + b$. Then taking $z = t^{-2/5}$ and $\psi \to t^{-3/10}\psi$, in parallel with Example 3.42, we end up with an equation[29]

$$\left(\frac{25}{4}(\zeta\partial_\zeta)^2 + \frac{25}{4}\zeta\partial_\zeta + \frac{21}{16} - (\partial_\zeta^2 + a\partial_\zeta^{6/5} + b\partial_\zeta^{4/5})\right)\mathcal{B}\psi(\zeta) = 0. \tag{3.105}$$

[29] The fractional derivative operator can be defined as follows. First, for $k < 0$ we can set

$$\partial_\zeta^k \phi(\zeta) = \frac{1}{\Gamma(-k)} \int_0^\zeta (\zeta - \zeta')^{-k-1} \phi(\zeta')\, \mathrm{d}\zeta' \tag{3.104}$$

In the case $a = b = 0$ things go just as they did in Example 3.42, with some replacements $\frac{2}{3} \to \frac{2}{5}$ (cf. Exercise 3.24.) More generally we have to grapple with the fractional derivatives. [find references for this material, to see whether what we say below is right!] The coefficients are singular only at $\zeta = \pm\frac{2}{5}$, so it seems likely the equation (3.105) still enables analytic continuation along generic rays to $\zeta = \infty$.

First suppose $a = 0$. In this case power counting suggests that there is only a regular singularity at $\zeta = \infty$, so the analytic continuation of $\mathcal{B}\psi^{(i)}$ should have moderate growth. Moreover the singularities at $\zeta = \pm\frac{2}{5}$ are also formally regular, and we may hope the analysis of monodromy around these points will be identical to what we had for the Airy equation.

For $a \neq 0$ the situation seems a bit more complicated since in this case all the singularities are formally irregular rather than regular. (Probably this is related to the fact that in this case the asymptotic exponent is not equal to t^{-1}, but has subleading corrections.)

[...]

3.11. ***k*-summation and multisummation.** See the excellent book [18] for a clear and efficient treatment of the material of this section.

Sometimes we encounter divergent series to which Borel summation cannot be applied directly. One example is a series whose coefficients grow faster than $n!$. A particular situation which often occurs (e.g. for solutions of differential equations) is that the coefficients grow like $\Gamma(1 + n/k)$ for some $k \in \mathbb{R}_+$.

One way to deal with this is to make a substitution $t \to t^{1/k}$ (and remember to undo this substitution at the end.) This is what we did above for the Airy equation, where we had $k = \frac{2}{3}$.

Another way to think about it is to keep the original variable but use a slight variant of Borel summation:

Definition 3.44 (*k*-summation). For a formal series f as usual we let

$$\mathcal{B}_k f(\zeta) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(1 + n/k)} \zeta^n \,. \tag{3.106}$$

If this converges in a disc and admits analytic continuation in a neighborhood of the ray $\mathbb{R}_+ t$, obeying a bound $|\mathcal{B}_k f(\zeta)| < \mathrm{e}^{M|\zeta|^k}$ as $\zeta \to \infty$, we say $f(t)$ is *k-summable* along this ray and write

$$\mathcal{S}_k f(t) = \mathcal{L}_k \mathcal{B}_k f(t) = \int_0^{t\infty} \mathrm{e}^{-(\zeta/t)^k} \mathcal{B}_k f(\zeta)\, \mathrm{d}(\zeta/t)^k \tag{3.107}$$

for t in the domain [...]

A useful rule of thumb is that we want to choose k in such a way that $\mathcal{B}_k f$ has a finite radius of convergence. (Of course, given such an k we could also choose any $k' < k$, in which case $\mathcal{B}_{k'} f$ will be entire. This might seem to be an advantage: no need to work hard to prove the analytic continuation. But the tradeoff is that in this situation it may be hard to show that $|\mathcal{B}_{k'} f(\zeta)| < \mathrm{e}^{M|\zeta|^{k'}}$ as $\zeta \to \infty$.)

To get $k > 0$ we then take the ordinary derivative sufficiently many times. The fractional derivative so defined has the expected behavior on polynomials, and hopefully also on convergent power series, so that the Borel transform indeed takes $t^{-k} \mapsto \partial_\zeta^k$ as needed below.

It may happen that $\mathcal{B}_k f$ does have a finite radius of convergence but does not admit analytic continuation along rays, or does not obey the necessary exponential bound. In this case f is not k-summable. However, suppose that $\mathcal{B}_k f$ obeys a weaker exponential bound, $|\mathcal{B}_k f(\zeta)| < \mathrm{e}^{M|\zeta|^\kappa}$ for some $\kappa > k$. In this case, let $k' > k$ be determined by $1/\kappa = 1/k - 1/k'$; then $\mathcal{B}_k f$ lies in the domain of the *acceleration operator* $\mathcal{A}_{k',k}$ defined by

$$\mathcal{A}_{k',k}\phi(\zeta') = \int_0^{\zeta'\infty} \phi(\zeta) C_{k'/k}((\zeta/\zeta')^k) \mathrm{d}(\zeta/\zeta')^k \tag{3.108}$$

where $C_\alpha(t)$ is the acceleration kernel

$$C_\alpha(t) = \frac{1}{2\pi \mathrm{i}} \int_\gamma t^{1/\alpha - 1} \mathrm{e}^{u - t u^{1/\alpha}} \mathrm{d}u \tag{3.109}$$

with γ the keyhole contour beginning and ending at $-\infty$ [fig]. This operator has

$$\mathcal{A}_{k',k}(\zeta^n) = \frac{\Gamma(1+n/k)}{\Gamma(1+n/k')} \zeta'^n. \tag{3.110}$$

Thus *formally* we have

$$\mathcal{L}_k = \mathcal{L}_{k'} \circ \mathcal{A}_{k',k}. \tag{3.111}$$

Now the hope is that the right side might be defined even when the left side is not, i.e. $\mathcal{A}_{k',k}\mathcal{B}_k f$ might admit analytic continuation along rays and obey an exponential bound so that $\mathcal{L}_{k'}\mathcal{A}_{k',k}\mathcal{B}_k f$ exists. In this situation we say that f is (k',k)-summable. Similarly we can define k-multisummability for any $k = (k_1, \dots, k_l)$ with $k_1 > k_2 > \cdots > k_l > 0$: the multisum is $\mathcal{L}_{k_1}\mathcal{A}_{k_1,k_2}\cdots\mathcal{A}_{k_{l-1},k_l}\mathcal{B}_{k_l} f$. Sometimes it is convenient to set $k_0 = \infty$ and think of $\mathcal{L}_{k_1}$ as being the acceleration operator $\mathcal{A}_{\infty,k_1}$; indeed $\lim_{\alpha\to\infty} C_\alpha(t) = \mathrm{e}^{-t}$. Note that at each stage of the process we have to do an analytic continuation and thus singularities may appear, giving rise to potential Stokes phenomena at each stage.

One simple way to get a (k,k')-multisummable function is to take the sum of a k-summable function and a k'-summable function. It is a general theorem that all multi-summable functions can be decomposed in this form, though not uniquely.

3.12. **Irregular Riemann-Hilbert correspondence.** Here is the key fact which motivated the definition of Stokes gradings: [refs: Braaksma, Balser close to our point of view; Martinet-Ramis for a different way, ...]

Construction 3.45 (Existence of Stokes gradings for irregular connections). There is a canonical functor

$$\mathrm{Sol} : \mathcal{C}_{\mathrm{dR}}(G, C, D, \Theta) \to \mathcal{C}_{\mathrm{Betti}}(G, \widehat{S}, \theta). \tag{3.112}$$

Proof. Fix $(\mathcal{F}, \nabla) \in \mathcal{C}_{\mathrm{dR}}(G, C, D, \Theta)$. As usual, we have the corresponding local system of solutions over S, which we can pull back to $\widehat{S}$. What remains is to give the Stokes gradings along the boundary.

We consider the bundle

$$\mathcal{E} = \mathrm{Hom}(\mathcal{F}, \mathcal{F}^\Theta) \tag{3.113}$$

which has a canonical connection $\nabla^* \otimes \nabla^{\Theta,m}$. To say that $(\mathcal{F}, \nabla)$ has asymptotic exponents Θ is to say that there exists a formal section g_{formal} of $\mathcal{E}$, covariantly constant away from the singularity. We want to promote g_{formal} to a meromorphic, invertible, covariantly constant section g of $\mathcal{E}$, obeying a Stokes condition. Then the desired Θ_ϕ-grading of $\mathcal{F}$ will be obtained by using g to pull back the canonical Θ_ϕ-grading of $\mathcal{F}^\Theta$.

We only give some indication of the construction of g, in the simplest situation. The asymptotic exponents of $(\mathcal{E}, \nabla^* \otimes \nabla^{\Theta,m})$ are the pairwise differences $q_i - q_j$. Take a local coordinate such that the singularity is at $z = 0$. Each $(q_i - q_j)_l = c_{ij} z^{\alpha_{ij}}$ for some $\alpha_{ij} \in \mathbb{Q}$. We consider the case where $\alpha_{ij} = \alpha$, independent of i, j; this is sometimes called the *one-level* case. Further assume all c_{ij} are distinct.

Then, one can prove that g_{formal} is $\frac{1}{\alpha}$-summable, with Stokes rays at $\arg \zeta = \frac{1}{\alpha} \arg c_{ij}$, i.e. $\zeta^\alpha / c_{ij} \in \mathbb{R}_+$. More precisely, $\mathcal{B}_\alpha(g_{\mathrm{formal}})$ has continuation up to singularities which occur at $\zeta^\alpha = c_{ij}$.

The very rough idea is as follows. We look at the equation obeyed by g_{formal},

$$g A_z^\Theta = A_z g + \partial_z g \,, \tag{3.114}$$

and then pass to the Borel transforms,

$$\mathcal{B}(g A_z^\Theta) = \mathcal{B}(A_z g) + \mathcal{B}(\partial_z g) \,. \tag{3.115}$$

This becomes an integral equation obeyed by $\tilde{g} = \mathcal{B}(g)$ (since $\mathcal{B}$ takes products to convolutions), and one obtains the desired summability by careful analysis of this equation. This argument can be found in [19].

In the case where multiple distinct exponents α_{ij} occur ("multi-level case") the situation is more complicated: then we do not expect g_{formal} to be summable, but it is multisummable, with the k's appearing given by the various $1/\alpha_{ij}$. This situation is also discussed in [19]. □

Theorem 3.46 (Irregular Riemann-Hilbert map is an equivalence). The functor Sol of Construction 3.45 is an equivalence.

Proof. This is parallel to the proof of Theorem 2.15. The only new ingredient we need is to show that, for a boundary circle ∂ and a system of local asymptotic exponents Θ, every Stokes graded local system over a neighborhood U of ∂ can indeed be realized by a meromorphic connection. This requires a construction; it is described in [19] Section 9.7. (In the case of Theorem 2.15 this part was easier: it just required us to take a logarithm of the monodromy.) [also need fullness: given a map between two Stokes graded local systems, we can lift it to a map of irregular connections, but need to see that this map is meromorphic around singularities]

□

[discuss a few examples] [explain role of formal monodromy as monodromy of the formal solutions]

4. The WKB method

Now we move on to trying to *compute* the Riemann-Hilbert map.

4.1. **WKB for Schrödinger equations.**

4.1.1. *Schrödinger equations with polynomial potential.* Suppose $P(z)$ is a polynomial of degree n. Then we consider the polynomial Schrödinger equation (1.38), repeated here:

$$\left(\partial_z^2 + P(z)\right)\psi(z) = 0. \tag{4.1}$$

(The Airy equation (1.12) is the special case $P(z) = -z$.) As we have discussed, the corresponding Stokes graded local system determines a point $[L] \in \mathrm{Conf}^{\mathrm{cyc}}_{n+2}(\mathbb{CP}^1)$. What is $[L]$?

4.1.2. *The WKB series.* To get a concrete handle on this, we will again "expand around infinity" but now in terms of a new parameter: we introduce $\hbar \in \mathbb{C}^\times$ and consider the equation

$$\left(\partial_z^2 + \hbar^{-2}P(z)\right)\psi(z) = 0. \tag{4.2}$$

So now our problem is to describe the family $[L_\hbar]$.[30] We will not give a closed form answer to this question, but we will explain how to construct an asymptotic series whose Borel summation gives $[L_\hbar]$ (in an appropriate sense), and we will understand its Stokes phenomena.

We need to get a handle on the local solutions of (4.2). A first try would be to write

$$\psi(z,\hbar) = \exp\left(\hbar^{-1}\int_{z_0}^{z}\lambda\right) \tag{4.3}$$

for some holomorphic 1-form $\lambda = \lambda_z\,\mathrm{d}z$. Plugging this into (4.2) and multiplying by $\hbar^2$ gives

$$(P(z) + \lambda_z^2) + \hbar\partial_z\lambda_z = 0. \tag{4.4}$$

To leading order in $\hbar$, this equation can be solved by setting

$$\lambda_z = \pm\sqrt{-P(z)}. \tag{4.5}$$

Note the ambiguity of the square root, which can be resolved locally but not globally. Said otherwise, globally this approximate solution lives, not on the base $C = \mathbb{CP}^1$, but rather on the Riemann surface

$$\Sigma = \{(y,z) : y^2 + P(z) = 0\} \subset \mathbb{C}^2. \tag{4.6}$$

Σ is a branched double cover of $\mathbb{C} \subset C$, with ramification at the n zeroes of $P(z)$ (counted with multiplicity). This surface will play an important role in the geometry of the WKB method.

Exercise 4.1. Prove that Σ is smooth except at the preimages of non-simple zeroes of $P(z)$. (In particular, if all zeroes of $P(z)$ are simple, then Σ is smooth everywhere.)

[30] Roughly this means we are trying to give a map $\mathbb{C}^\times \to \mathrm{Conf}^{\mathrm{cyc}}_{n+2}(\mathbb{CP}^1)$. But more precisely it is a twisted map, i.e. a section of a bundle with fiber $\mathrm{Conf}_{n+2}(\mathbb{CP}^1)$, because when $\arg\hbar$ varies by an angle ϑ, the Stokes rays rotate by $\frac{2}{n+2}\vartheta$, cf. Exercise 3.6.

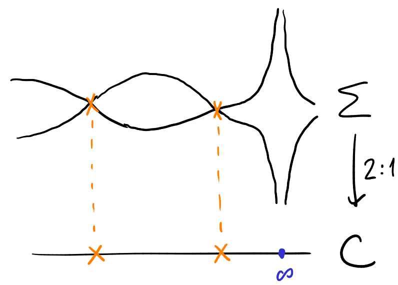

FIGURE 18. The Riemann surface Σ, shown with its projection to C. This projection is a branched double cover of $\mathbb{C} \subset C$, but misses the point $\infty \in C$. Zeroes of $P(z)$ are branch points of the covering; they are shown as orange crosses on C. Their preimages are ramification points; they are shown as orange crosses on Σ.

We note a few other facts about this surface:

- Σ carries a natural holomorphic 1-form
$$\lambda_0 = y\,\mathrm{d}z. \tag{4.7}$$
- If z_0 is a simple zero of $P(z)$, then λ_0 has a double zero at the corresponding ramification point $(0, z_0) \in \Sigma$. (Indeed, in a neighborhood of $(0, z_0)$, Σ is locally coordinatized by y, and $y\,\mathrm{d}z = y\frac{\mathrm{d}z}{\mathrm{d}y}\mathrm{d}y = -\frac{2y^2}{P'(z)}\mathrm{d}y$.)
- Σ can be thought of as an extension of the covering Θ from the boundary circle ∂ to the whole surface $\widehat{S}$.

Now, (4.3) is not an actual solution, except if $P(z)$ is constant: there is the uncancelled term $\hbar\partial_z\lambda_z$ in (4.4). In order to do better we attempt to write a solution in the more general form
$$\psi(z, \hbar) = \exp\left(\hbar^{-1}\int_{z_0}^{z}\lambda(\hbar)\right). \tag{4.8}$$
Requiring that this solves (4.2) means that $\lambda(\hbar)$ is a solution of the Riccati equation
$$\lambda_z^2 + P(z) + \hbar\partial_z\lambda_z = 0. \tag{4.9}$$
We can determine such a λ (defined on Σ) order by order in $\hbar$: we write
$$\lambda_{\text{formal}} = \sum_{n=0}^{\infty}\lambda_n\hbar^n \tag{4.10}$$
and fix the first-order part as $\lambda_0 = y$; then (4.9) requires
$$\lambda_{\text{formal},z} = y - \hbar\frac{P'}{4P} + \hbar^2 y\frac{5P'^2 - 4PP''}{32P^3} + \cdots \tag{4.11}$$

Exercise 4.2. Verify the expansion (4.11) and work out the next term.

Note that although (4.9) is a differential equation for λ, the formal solution λ_{formal} is computed purely algebraically, without doing any integrals. The reason for this is that the equation is algebraic *to leading order in* $\hbar$, so at each order we only meet derivatives of the terms already computed at previous orders.

4.1.3. *An obstruction to Borel summation of the WKB series.* Now, the key question is: can λ_{formal} be summed into an actual 1-form λ?

We can ask this question for each $z \in C$ separately: can $\lambda_{\text{formal}}(z)$ be summed to a cotangent vector $\lambda(z) \in T_z^*C$? A preliminary remark is that, when $P(z)$ has at least one simple zero, one can see directly that the answer cannot be "yes, for all z".

Indeed, assume that for some fixed $\hbar \in \mathbb{C}^\times$ there exists a global 1-form λ on Σ obeying (4.9). In that case, (4.8) would give a global solution $\psi(z,\hbar)$ to (4.2), defined on Σ. Pushing $\psi(z,\hbar)$ forward would give two local solutions $\psi^{(i)}(z,\hbar)$ on C. Monodromy would exchange the $\psi^{(i)}$ when z goes around a simple zero of $P(z)$. On the other hand, the zeroes of $P(z)$ are perfectly regular points for the equation (4.2); the monodromy around them has to be trivial. This contradiction shows that our assumption must have been wrong: there must be a failure of Borel summability occurring on any loop that goes around a zero of $P(z)$.

(This might remind you of our proof that nontrivial Stokes matrices have to appear in the analysis of solutions of the Airy equation around $z = \infty$. Indeed, for the rescaled Airy equation, the expansion around $z = \infty$ and around $\hbar = 0$ are essentially the same, as one can see by a change of variables. This is an exceptional coincidence, which doesn't occur for more general equations.)

4.1.4. *Trajectories of a quadratic differential.* To describe the Stokes phenomena precisely, we need a little bit of geometry. We formulate it in a slightly more general context, which will be useful later.

First we define:

Definition 4.1 (Spectral curve for a quadratic differential). If C is a Riemann surface and ϕ_2 is a meromorphic quadratic differential on C, then the *spectral curve* is

$$\Sigma = \{(s,z) : s^2 + \phi_2(z) = 0\} \subset T^*C. \tag{4.12}$$

It carries a canonical involution $\sigma : (s,z) \mapsto (-s,z)$.

Exercise 4.3. Prove that Σ is smooth except at the preimages of non-simple zeroes of ϕ_2. In particular, if all zeroes of ϕ_2 are simple, then Σ is smooth everywhere. (Hint: use a local coordinate system to reduce this to Exercise 4.1.)

For any manifold M there is a canonical 1-form λ_0 on T^*M, sometimes called the Liouville 1-form. Its key property is that

$$\mathrm{d}\lambda_0 = \omega \tag{4.13}$$

where ω denotes the standard symplectic form on T^*M. In local coordinates (p^i, q_i) on T^*M induced by coordinates (q_i) on M, one has

$$\lambda_0 = \sum_i p^i \, \mathrm{d}q_i. \tag{4.14}$$

Exercise 4.4. Prove that λ_0 defined by (4.14) is independent of the choice of local coordinate patch on M. (One nice way to do this is to give a coordinate-independent definition of λ_0.)

Similarly, if X is a complex manifold, then there is a canonical holomorphic Liouville form λ_0 on T^*X, again given locally by (4.14) where now (q_i) are local holomorphic coordinates.

Definition 4.2 (Canonical 1-form). The canonical 1-form λ_0 on $\Sigma \subset T^*C$ is the pullback of the holomorphic Liouville form from T^*C to Σ.

Example 4.3 (Spectral curves for polynomial potentials). If we specialize to $C = \mathbb{CP}^1$, and fix a polynomial $P(z)$ of degree n, then we can consider

$$\phi_2 = P(z)\,\mathrm{d}z^2\,. \tag{4.15}$$

ϕ_2 is a meromorphic quadratic differential on $C = \mathbb{CP}^1$, which has a pole of order $n+4$ at $z = \infty$. Then Definition 4.1 agrees with our previous definition (4.6), with the translation $s = y\,\mathrm{d}z$. Likewise, Definition 4.2 agrees with our previous definition (4.7).

Definition 4.4 (Action coordinate). Given a simply connected domain $U \subset \Sigma$ we consider a map $w : U \to \mathbb{C}$ obeying the condition

$$\mathrm{d}w = \lambda_0. \tag{4.16}$$

If λ_0 has no zeroes on U, then w gives a coordinate on U (possibly after shrinking U). This coordinate is canonical up to a constant shift $w \to w + c$.

Definition 4.5 (ϑ-trajectories on Σ). A ϑ-trajectory on Σ is a path on Σ along which $\operatorname{Im} \mathrm{e}^{-\mathrm{i}\vartheta} w$ is constant. It is canonically oriented in the direction in which $\operatorname{Re} \mathrm{e}^{-\mathrm{i}\vartheta} w$ decreases.

Said otherwise, ϑ-trajectories are straight line segments in the w-coordinate, of inclination ϑ, with orientation reversed. Note that this definition is canonical even though the coordinate w has an ambiguity, since a shift $w \to w + c$ preserves the property of being a straight line with inclination ϑ.

Exercise 4.5. If b is a ϑ-trajectory on Σ, show that $\arg(-\int_b \lambda_0) = \vartheta$.

The ϑ-trajectories on Σ make up a foliation of $\Sigma^\circ = \{p \in \Sigma : \lambda_0(p) \neq 0\}$. The action of σ maps leaves to leaves, but reverses the orientation.

Definition 4.6 (ϑ-trajectories on C). A ϑ-trajectory on C is a path which is the projection of a ϑ-trajectory on Σ.

The ϑ-trajectories on C make up a foliation of $C^\circ = \{p \in C : \phi_2(p) \neq 0\}$.

4.1.5. *Behavior around the zeroes.* As we have discussed, λ_0 has zeroes at the ramification points of Σ, lying over the zeroes of ϕ_2. The behavior of the foliation around such a zero is:

Proposition 4.7 (Behavior of ϑ-foliation of Σ, around a zero of λ_0). Up to a conformal map, the ϑ-foliation around a zero of λ_0 of order k is as shown below.

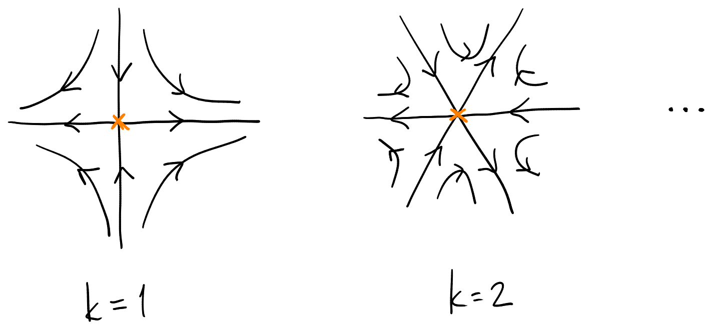

FIGURE 19. The ϑ-foliation on Σ, in a neighborhood of a zero of λ_0, of order k.

Proof. We can find a local coordinate t around the zero such that $e^{-i\vartheta}\lambda_0 = -t^k dt$. Then $e^{-i\vartheta}w(t) = \frac{t^{k+1}}{k+1}$, and pulling back horizontal lines by this map gives the pictures shown. $\square$

For our purposes the most important figure above is the one for $k = 2$, since this is the behavior around a simple ramification point of Σ. Note in particular that there exist ϑ-trajectories which pass through the ramification point, in 3 distinguished directions.

Proposition 4.8 (Behavior of ϑ-foliation of Σ, around a higher-order pole of λ_0). Up to a conformal map, the ϑ-foliation around a pole of λ_0 of order $-k > 1$ is as shown below. In particular, there are $-2k - 2$ distinguished directions around the pole, of which half are attracting and half are repelling.

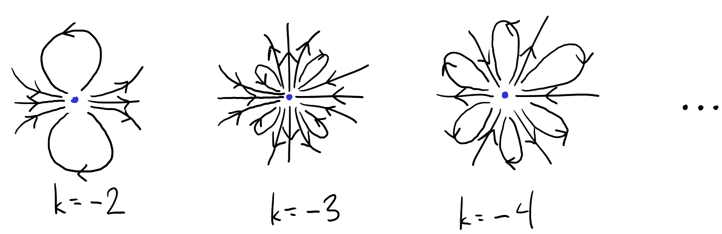

FIGURE 20. The ϑ-foliation on Σ, in a neighborhood of a pole of λ_0, of order $-k > 1$.

Proof. As in Proposition 4.7: pull back the foliation by horizontal lines through the map $e^{-i\vartheta}w(t) = \frac{t^{k+1}}{k+1}$. $\square$

The case of a simple pole is a bit more interesting, because there is a local invariant, the residue.

Proposition 4.9 (Behavior of ϑ-foliation of Σ, around a simple pole of λ_0). Let m denote the residue of λ_0 at a simple pole. Then up to a conformal map, the ϑ-foliation around the pole is as shown below.

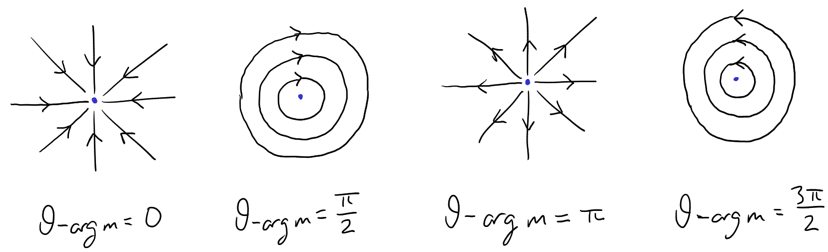

FIGURE 21. The ϑ-foliation on Σ, in a neighborhood of a simple pole of λ_0.

Proof. Fix a coordinate in which $\lambda_0 = m\frac{dt}{t}$. Then we may take $w = m \log t$, i.e. $t = e^{w/m}$. Then pulling back straight lines of inclination ϑ from the w-plane, and reversing, gives the pictures above. □

It will be convenient also to draw the corresponding pictures on the base C.

Proposition 4.10 (Behavior of the ϑ-foliation of C, around a simple zero of ϕ_2). Up to a conformal map, the ϑ-foliation around a simple zero of ϕ_2 is as shown below.

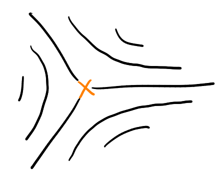

FIGURE 22. The ϑ-foliation on C, in a neighborhood of a simple zero of the quadratic differential ϕ_2.

Proof. Here are two methods:

- Take the quotient of the $k = 2$ case of in Figure 19 by the action of σ, which acts by $t \to -t$.
- Adopt a local coordinate on C in which $\phi_2 = -z\, dz^2$; then consider a double-valued function on C,

$$w = \int_0^z \sqrt{-\phi_2} = \int_0^z \sqrt{z'}\, dz' = \frac{2}{3} z^{3/2}. \tag{4.17}$$

  Pulling back straight lines in the w-coordinate to the z-plane then gives the picture shown. □

Note that the ϑ-foliation on C is unoriented; indeed the action of σ maps each leaf segment to a leaf segment with the opposite orientation, so at least there is no orientation induced from the orientation of the leaves on Σ; moreover one can quickly convince oneself that there is no way to continuously orient the leaves shown in Figure 22.

Exercise 4.6. Describe the behavior of the ϑ-foliation on C in a neighborhood of an order k zero of ϕ_2. Show that there are $k + 2$ ϑ-trajectories ending on the zero, generalizing Figure 22 in the $k = 1$ case. Similarly show that if ϕ_2 has a first-order pole then there is a unique ϑ-trajectory ending on the pole.

Definition 4.11 (Critical ϑ-trajectories). A ϑ-trajectory on C which has one end on a zero or first-order pole of ϕ_2 is called a *critical ϑ-trajectory*. The union of all critical ϑ-trajectories on C is called the *critical ϑ-graph* $\mathcal{W}(\vartheta)$.

[should redefine this so $\mathcal{W}(\vartheta)$ is on Σ instead of C, to match better with later sections]

Exercise 4.7. Describe the behavior of the ϑ-foliation on C in a neighborhood of an order $-k$ pole of ϕ_2, with $-k > 2$. Show that there are $-k-2$ special directions to which the trajectories asymptote as they approach the pole.

Exercise 4.8. Describe the behavior of the ϑ-foliation on C in a neighborhood of an order 2 pole of ϕ_2. (Hint: it depends on the residue, similarly to Proposition 4.9.)

4.1.6. *Behavior in the large.* We continue to hold C and ϕ_2 fixed. The global behavior of the ϑ-trajectories is well understood thanks to theorems from [20], as we now describe.

Definition 4.12 (Length). The *length* of a path b on C (with respect to ϕ_2) is $\int_b |\sqrt{\phi_2}| \in \mathbb{R}$.

Equivalently, the length of b is the length of the image of b in the w-plane. Also equivalently, it is the length of b with respect to the singular flat Riemannian metric $|\phi_2|$ on C.

Definition 4.13 (Finite ϑ-trajectories). A ϑ-trajectory is called *finite* if it is maximal (not properly contained in another ϑ-trajectory) and has finite length.

The finite ϑ-trajectories will turn out to be particularly important: they are responsible for the Stokes phenomena in the asymptotics of $[L_\hbar]$. With this in mind we spend some time on them now.

Proposition 4.14 (Classification of finite trajectories). A finite ϑ-trajectory is either a *saddle connection* (interval, with both ends on zeroes or first-order poles of ϕ_2, possibly the same one at both ends) or a *loop* (trajectory with the topology of S^1).

Proof. [...] □

Definition 4.15 (Charge of a finite trajectory). The charge of a finite trajectory is a class $\gamma \in H_1(\Sigma^\circ, \mathbb{Z})$ defined as follows. [...]

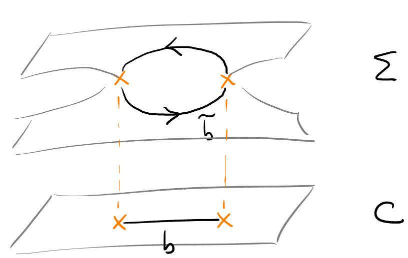

FIGURE 23. The charge of a saddle connection.

Proposition 4.16 (Phases of periods of finite trajectories). If γ is the charge of a finite ϑ-trajectory, then $\arg(-Z_\gamma) = \vartheta$.

Proof. This follows directly from Exercise 4.5. □

Definition 4.17 (BPS phases). We call ϑ a *BPS phase* for ϕ_2 if there exists a finite ϑ-trajectory.

Corollary 4.18 (BPS phases are countable). The set of BPS phases for ϕ_2 is countable.

Proof. Every finite ϑ-trajectory has a charge γ. The periods Z_γ for $\gamma \in H_1(\Sigma, \mathbb{Z})$ take values in a countable set. Now use Proposition 4.16. □

It follows that a generic phase ϑ is non-BPS. Loosely speaking: as we vary ϑ through non-BPS phases, the topology of $\mathcal{W}(\vartheta)$ does not change; when we cross a BPS phase, the topology of $\mathcal{W}(\vartheta)$ does change, in a way dictated by the nature of the finite trajectory.

Example 4.19 (Trajectory structure for polynomial quadratic differentials). We continue with the setup of Example 4.3. Suppose

$$P(z) = az^n + \cdots . \tag{4.18}$$

First we describe the asymptotic behavior of the ϑ-trajectories near $z = \infty$. We work on Σ, which means we treat $\sqrt{z}$ as a good coordinate and also choose a branch of $\sqrt{-a}$. The asymptotic behavior of w near $z = \infty$ is

$$w \approx \int_{z_0}^{z} \sqrt{-a} z'^{n/2} \mathrm{d}z' = \frac{2}{n+2} \sqrt{-a} z^{n/2+1} + c. \tag{4.19}$$

ϑ-trajectories asymptotically approach the rays with $\arg w = \vartheta$, i.e.

$$(n+2) \arg \sqrt{z} = \vartheta - \arg \sqrt{-a}\,. \tag{4.20}$$

This gives $n+2$ asymptotic rays on Σ, located at

$$\arg \sqrt{z} = \frac{\vartheta - \arg \sqrt{-a}}{n+2}\,. \tag{4.21}$$

Projecting these down to C gives $n+2$ asymptotic rays going into $z = \infty$.

Using results from [20] one can show that a ϑ-trajectory on C either goes asymptotically out to $z = \infty$ in one of these $n+2$ directions, or ends on one of the zeroes of $P(z)$. If it has both ends on zeroes of $P(z)$, it is a saddle connection with phase ϑ. These can only occur if ϑ is a BPS phase. Thus, by Corollary 4.18, for generic ϑ, no saddle connections with phase ϑ exist.

Now suppose $P(z)$ has only simple zeroes. In this case, the critical graph consists of $3n$ trajectories, emitted from the zeroes and radiating out to the $n+2$ asymptotic rays. See Figure 24 for a few particularly symmetric examples, and Figure 25 for a more generic one.

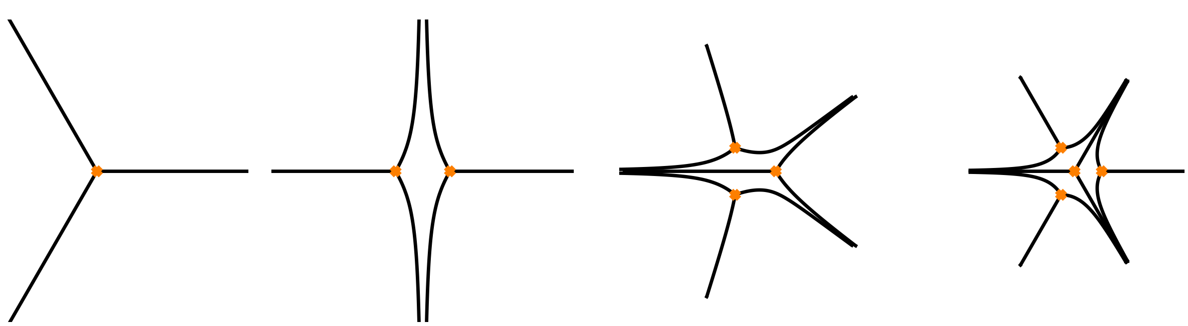

FIGURE 24. Critical graphs $\mathcal{W}(\vartheta = 0)$ for the potentials $P(z) = -z, -z^2 + 1, z^3 - 1, -z^4 + z$.

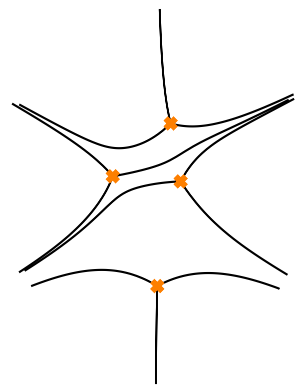

FIGURE 25. Critical graph $\mathcal{W}(\vartheta = 0)$ for the potential $P(z) = z^4 + (3 + \mathrm{i})z^3 + (6 + 2\mathrm{i})z^2 + 7z + 1$.

Exercise 4.9. Write a program which can reproduce these pictures, using your favorite programming language. Hint: it is tempting to write the trajectories as solutions $z(s)$ of the ODE $z'(s) = -\mathrm{e}^{-\mathrm{i}\vartheta}\sqrt{-P(z(s))}$, and use a numerical ODE solver. This runs into a difficulty: if you use the $\sqrt{\cdot}$ function provided by your standard library, the branch cut will cause problems. What you want instead is to have a continuous choice of $y(s)$ which obeys $y(s)^2 + P(z(s)) = 0$. For this purpose there is a nice trick: make an appropriate initial choice of $(y(0), z(0))$ obeying $y(0)^2 + P(z(0)) = 0$, and then apply the ODE solver to the coupled system

$$z'(s) = -\mathrm{e}^{-\mathrm{i}\vartheta}y(s), \qquad y'(s) = \frac{P'(z(s))z'(s)}{2y(s)}. \tag{4.22}$$

4.1.7. *Solitons and WKB.* In a sense we will try to make precise, the singularities of the Borel transform for $\arg\hbar = \vartheta$ propagate along ϑ-trajectories. This is the main reason why these trajectories are important for us.

Definition 4.20 (WKB solitons and periods). A *WKB soliton with phase ϑ over z* is a ϑ-trajectory b on Σ which begins and ends at distinct points of $\pi^{-1}(z)$. The *period* of the soliton is

$$Z_b = \int_b \lambda_0 \in \mathbb{C}\,. \tag{4.23}$$

Exercise 4.10. Say Z is the period of a WKB soliton with phase ϑ. Prove that $\arg(-Z) = \vartheta$.

Given a WKB soliton b with phase ϑ, its orientation reversal $-b$ is also a WKB soliton, with phase $\vartheta + \pi$.

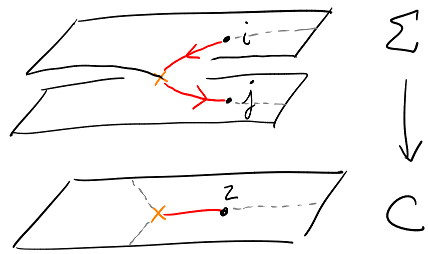

FIGURE 26. A WKB soliton at z. The critical graph $\mathcal{W}(\vartheta)$ is shown in dashed gray lines.

Figure 26 suggests the following proposition.

Proposition 4.21 (Critical graph and WKB solitons). $z \in \mathcal{W}(\vartheta)$ just if there exists a WKB soliton with phase ϑ over z.

Proof. [...] □

For a given $z \in C$ we will be interested in knowing the periods of all the WKB solitons over z.

Example 4.22 (WKB solitons for the Airy equation). If $P(z) = -z$, then for any $z \neq 0$ there are exactly 2 WKB solitons over z, with periods $\pm\frac{4}{3}z^{3/2}$, and phases $\vartheta = \arg(\pm z^{3/2})$.

If $P(z)$ is a more general polynomial, then the number of WKB solitons over z will typically depend on z, and there may be multiple WKB solitons b, b' with the same initial and final points i, j. Given two such WKB solitons, the concatenation $b - b'$ is a closed path on Σ, which then has a class $\gamma = [b - b'] \in H_1(\Sigma, \mathbb{Z})$. Thus the periods differ by $Z_b(z) - Z_{b'}(z) = Z_\gamma$ which is a locally constant function of z. We see this in the next example.

Example 4.23 (WKB solitons for the Weber equation). We can see an example of this already for the Weber equation, $P(z) = -z^2 + 1$. In this case there are either 2 or 4 WKB solitons over z, depending on the value of z.

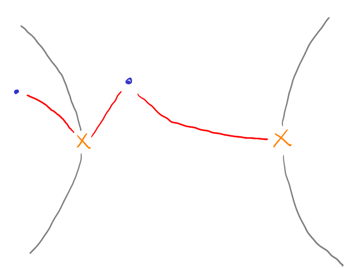

FIGURE 27. The projections of the WKB solitons to C (red arcs): there are either 1 or 2 depending on the position of z (blue point). The gray arcs mark the locus where the number of WKB solitons jumps.

In either case, the periods are integrals of the form $\int_{\pm 1}^{z} \sqrt{z'^2 - 1}\,\mathrm{d}z'$, which can be expressed in terms of elementary functions. In the interior region, for each choice of i, there are 2 distinct solitons b, b' running from i to j. They differ by a class $\gamma = [b - b'] \in H_1(\Sigma, \mathbb{Z})$, with period $Z_\gamma = 2\int_{-1}^{1} \sqrt{z^2 - 1}\,\mathrm{d}z = \pi \mathrm{i}$.

Exercise 4.11. Calculate the 4 periods explicitly for the points $z = 0$ and $z = \mathrm{i}$. Calculate the 2 periods explicitly for the point $z = 2$.

The transition from 4 to 2 WKB solitons occurs by degeneration: as z is varied, a long WKB soliton over z degenerates ("decays") into the union of a shorter WKB soliton and a saddle connection.

Exercise 4.12. Give a sketch proof that the gray arcs in Figure 27 are ϑ-trajectories, with phase $\vartheta = \frac{\pi}{2}$. (Hint: when a WKB soliton decays, it must have the same phase as its decay constituents; thus in particular its constituents must have the same phase.)

Exercise 4.13. Let C be any compact Riemann surface and ϕ_2 any meromorphic quadratic differential on C. For fixed $z \in C$, prove that the periods of WKB solitons over z form an at most countable subset of $\mathbb{C}$.

Conjecture 4.1 (Analytic continuation of Borel transform of the WKB series for polynomial potentials). When $\lambda_{\mathrm{formal},z}$ is the WKB series (4.11) associated to an equation (4.2), the Borel transform $\mathcal{B}\lambda_{\mathrm{formal},z}(\zeta)$ at $p \in \Sigma$ admits analytic continuation to the largest star-shaped region around 0 which does not contain any periods of WKB solitons beginning at p, and obeys an exponential bound $|\mathcal{B}\lambda_{\mathrm{formal},z}(\zeta)| < \mathrm{e}^{M|\zeta|}$ for every $M > 0$.

Although I call this a conjecture, parts are proven already. The analytic continuation and exponential bound for some M have been folklore for a long time, attributed to a manuscript of Koike-Schäfke; an extended proof sketch appeared in [21]. The basic idea is to pass to the coordinate w and there analyze the Borel transform of the Riccati equation. A more complete account of this has appeared recently in [22]. The statement that the singularities of $\mathcal{B}\lambda_{\mathrm{formal},z}$ are precisely at the periods of WKB solitons is essentially in [23], with some ideas of proof; I understand [check] that it is also in [24]. I am not sure whether anyone has proven that the exponential bound holds for *every* M, so that $\lambda(\hbar)$ really exists for all $\hbar$, not only for small enough $|\hbar|$; at least numerical experimentation is consistent with this.

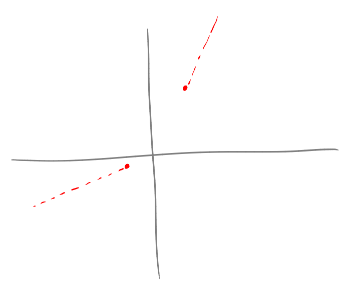

FIGURE 28. The slit plane where $\mathcal{B}\lambda_{\mathrm{formal},z}(\zeta)$ is defined. In this example there are 4 WKB solitons at z, of which two begin at i.

Corollary 4.24 (Borel summability of the WKB series for polynomial potentials). $\lambda_{\mathrm{formal},z}(\hbar)$ can be Borel summed at $p \in \Sigma$ to give a function $\lambda_z(\hbar)$, except when $\arg \hbar$ is the phase of a WKB soliton beginning at p.

4.1.8. *Stokes gradings via WKB.* Now fix $\hbar$ and a simply connected $U \subset C$ not meeting the critical graph $\mathcal{W}(\arg \hbar)$. Over U fix a trivialization $\Sigma \to \underline{2}$, given by two maps $s^{(i)} : C \to \Sigma$. Then write $\lambda^{(i)} = (s^{(i)})^* \lambda$. This gives two holomorphic 1-forms on U. If we also fix $z_0 \in U$, then we get two canonical solutions of (4.2),

$$\psi^{(i)}(z) = \exp \hbar^{-1} \int_{z_0}^{z} \lambda^{(i)}(\hbar, z) \,. \tag{4.24}$$

More abstractly: for fixed $\hbar$, let L be the local system of solutions of (4.2). If $z \in C$ does not lie on $\mathcal{W}(\arg \hbar)$, the solutions $\psi^{(i)}$ give a Σ_z-grading of the stalk L_z. Moreover, for $z \in \mathcal{W}(\arg \hbar)$ we have a partial ordering on Σ_z: namely, if there is a WKB soliton with phase ϑ running from j to i then $i \prec j$. In this situation Corollary 4.24 says $\psi^{(i)}$ is Borel summable at z, while $\psi^{(j)}$ is not. Thus the gradings on the two sides of the trajectory in $\mathcal{W}(\arg \hbar)$ containing z obey the Stokes condition (Definition 3.17) with respect to this ordering.

The structure we have found can be summarized as follows:

Definition 4.25 (Stokes graded local system, again). A *Stokes graded local system* over (C, ϕ_2) with phase ϑ is a local system L over C, equipped with a Σ_z-grading for each $z \in C \setminus \mathcal{W}(\vartheta)$, such that Σ_z varies in a locally constant way, and when $\Sigma_z \in \mathcal{W}(\vartheta)$ the gradings of L_z coming from the two sides of z obey the Stokes condition.

Construction 4.26 (WKB gives Stokes gradings). The local system $L_\hbar$ of solutions to (4.2) admits a natural Stokes grading G over (C, ϕ_2) with phase $\vartheta = \arg \hbar$; the local solution (4.24) gives a section of $G^{(i)}$. Call this Stokes grading the *WKB grading*.

There is an important compatibility between the $\hbar \to 0$ asymptotics and the $z \to \infty$ asymptotics. Note that in a neighborhood of $z = \infty$ we can identify the double cover Σ with the double cover Θ giving the asymptotic exponents for the Schrödinger equation. Then we have two different Stokes gradings on $L_\hbar$ over the boundary of the disc $\widehat{S}$:

- the one given by general theory of expansion around irregular singularities in Construction 3.45,
- the WKB grading of Construction 4.26.

Conjecture 4.2 (Compatibility of WKB and irregular singularity asymptotics). These two Stokes gradings agree.

This is again a kind of well-known folklore; I list it as a conjecture to be safe. I would be happy to learn of a reference.

4.1.9. *Abelianization.*

Construction 4.27 (Abelianization). Suppose given a Stokes graded local system L over (C, ϕ_2) with phase ϑ. We build a rank 1 local system A over Σ°, as follows. Away from $\mathcal{W}(\vartheta)$, we define A on sheet i to be $\pi^* G^{(i)}$. Now consider a trajectory (wall) in $\mathcal{W}(\vartheta)$, with $i \prec j$. The gradings on the two sides have $G^{(i)} = G'^{(i)}$. Thus we can extend A straightforwardly over the wall on sheet i. On sheet j, it seems A does not extend directly, since $G^{(j)} \neq G'^{(j)}$. But we have the canonical Stokes automorphism $S^{G \to G'} : G^{(j)} \to G'^{(j)}$, and we can use this to glue. We call A the *abelianization* of L.

Proposition 4.28 (Abelianization gives an almost local system over Σ). If A is the abelianization of L, the holonomy of A around a simple ramification point is -1.

Proof. We consider the local picture in a neighborhood $U \subset C$ as shown below.

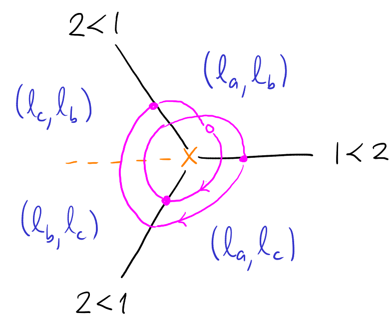

FIGURE 29. A neighborhood $U \subset C$ of a branch point, and a loop p on Σ which goes once around the ramification point. The initial point of p is marked with a hollow dot. p crosses each wall twice, once on each sheet; on a wall labeled $i \prec j$, we mark with a solid dot the crossing on sheet j. In each domain, the Σ-grading of L is shown; e.g. in the northeast domain we have $(G^{(1)}, G^{(2)}) = (\ell_a, \ell_b)$.

Let V be the 2-dimensional vector space $L(U)$. In each domain of the figure, the WKB Σ-grading gives a decomposition of V into two lines, indexed by the two sheets of Σ. *A priori* this would mean we have six lines (two in each of the three domains), but the Stokes condition imposes relations, so ultimately there are only three lines $\ell_{a,b,c}$, as indicated in the figure. We choose elements $\psi_{a,b,c}$ in $\ell_{a,b,c}$ respectively.

Now, the element $\psi_a \in \ell_a$ gives a section of A on sheet 1 in the northeast domain of the figure. We compute the continuation of ψ_a around the path p shown in the figure. Following the definition of A, this continuation agrees with the continuation in L, except when the path lies on sheet j and crosses a wall labeled $i \prec j$; at each wall, the continuation is given by the corresponding Stokes matrix. Along the path p there are three such crossings, and the corresponding Stokes matrices map (using Exercise 3.10)

$$\psi_a \in \ell_a \mapsto \frac{\psi_a \wedge \psi_c}{\psi_b \wedge \psi_c}\psi_b \in \ell_b, \qquad \psi_b \in \ell_b \mapsto \frac{\psi_b \wedge \psi_a}{\psi_c \wedge \psi_a}\psi_c \in \ell_c, \qquad \psi_c \in \ell_c \mapsto \frac{\psi_c \wedge \psi_b}{\psi_a \wedge \psi_b}\psi_a \in \ell_a\,. \tag{4.25}$$

Composing these three maps gives $\psi_a \mapsto -\psi_a$, so the total holonomy around p is -1 as desired.

We can visualize this composition of Stokes automorphisms as shown:

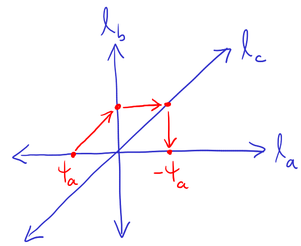

FIGURE 30. The composition of three Stokes automorphisms mapping $\ell_a \to \ell_b \to \ell_c \to \ell_a$ within the 2-dimensional vector space V.

□

[add exercise on realization of wedge products as Wronskians]

The next proposition shows that A contains at least some information about the local system L we started with. (Later we will see how to reconstruct L completely from A.)

Proposition 4.29 (Abelian holonomies compute cross-ratios). Suppose A is the abelianization of L. Consider a domain $U \subset C$ containing two branch points as indicated in Figure 31.

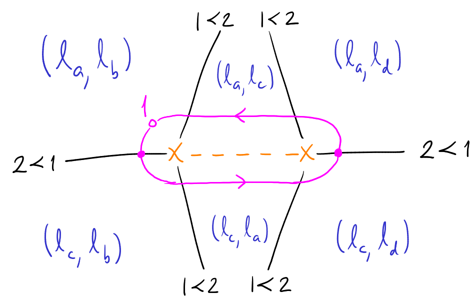

FIGURE 31. A domain $U \subset C$ containing two branch points. We show the line decompositions of $V = L(U)$ in all domains, involving four lines $\ell_{a,b,c,d}$. We also show a loop p on Σ.

The holonomy of A around the loop $p \subset \Sigma$ is the cross ratio of these four lines,

$$X_p = r(\ell_a, \ell_b, \ell_c, \ell_d) = \frac{(\ell_a \wedge \ell_b)(\ell_c \wedge \ell_d)}{(\ell_b \wedge \ell_c)(\ell_d \wedge \ell_a)} . \tag{4.26}$$

Proof. We follow the same pattern as in the proof of Proposition 4.28. We choose $\psi_{a,b,c,d} \in \ell_{a,b,c,d}$ and evaluate the continuation of ψ_a around p. There are two crossings involving nontrivial Stokes matrix elements, which give maps $\ell_a \to \ell_c \to \ell_a$,

$$\psi_a \mapsto \frac{\psi_a \wedge \psi_b}{\psi_c \wedge \psi_b} \psi_c , \qquad \psi_c \mapsto \frac{\psi_c \wedge \psi_d}{\psi_a \wedge \psi_d} \psi_a . \tag{4.27}$$

Composing these gives the desired (4.26). □

Example 4.30 (A cubic example). Let's consider the situation arising from the polynomial $P(z) = z^3 - 1$, and $\vartheta = 0$. The critical graph is as shown in Figure 32.

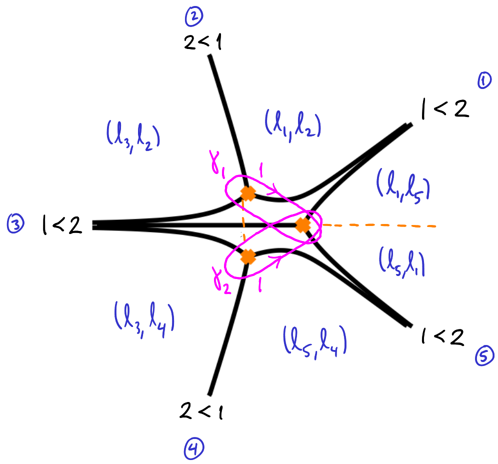

FIGURE 32. The critical graph associated to $P(z) = z^3 - 1$, at $\vartheta = 0$. We also show two cycles γ_1, γ_2 on the double cover Σ.

The two cycles γ_1, γ_2 form a basis of $\Gamma = H_1(\Sigma, \mathbb{Z})$. The corresponding holonomies of A are

$$\mathcal{X}_{\gamma_1} = r(\ell_2, \ell_3, \ell_5, \ell_1), \quad \mathcal{X}_{\gamma_2} = r(\ell_3, \ell_4, \ell_5, \ell_1). \tag{4.28}$$

We have five Stokes rays at infinity and $\mathcal{M}_{\mathrm{Betti}}(G, \widehat{S}, \theta) = \mathrm{Conf}_5(\mathbb{CP}^1)$. Note that the two cross-ratios $\mathcal{X}_{\gamma_1}, \mathcal{X}_{\gamma_2}$ together form a coordinate system on an open dense subset of $\mathcal{M}_{\mathrm{Betti}}(G, \widehat{S}, \theta)$.

For each non-BPS phase ϑ, the Stokes graph $\mathcal{W}(\vartheta)$ is dual to a triangulation of the disc $\widehat{S}$, with vertices at the Stokes rays. See Figure 33.

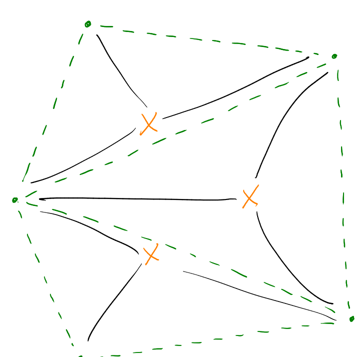

FIGURE 33. The triangulation of $\widehat{S}$ corresponding to $P(z) = z^3 - 1$ and $\vartheta = 0$. We represent the circular boundary of $\widehat{S}$ as a pentagon.

Each edge E of the triangulation corresponds to a cycle $\gamma_E \in H_1(\Sigma, \mathbb{Z})$, and the holonomy $\mathcal{X}_{\gamma_E}$ of A around γ_E is a cross-ratio of the four lines associated to the four vertices of the quadrilateral abutting E, as in Proposition 4.29. Now a natural question arises: how do these holonomies change when the triangulation changes?

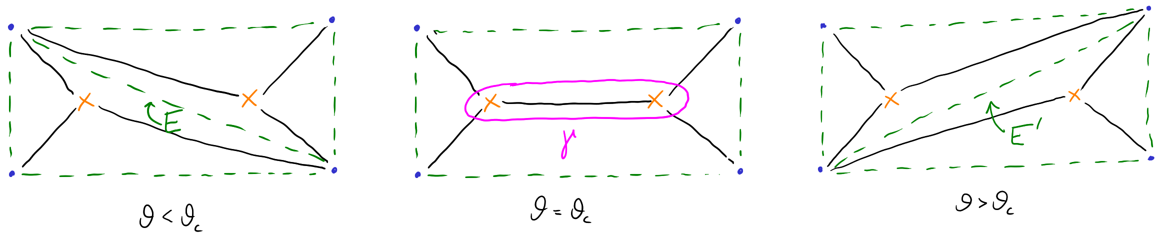

FIGURE 34. A flip of the triangulation, induced by the appearance of a saddle connection. We do not show the sheet labels explicitly; the class γ is determined by the condition $e^{-i\vartheta_c} Z_\gamma < 0$.

Proposition 4.31 (Stokes factor associated to a flip). Consider the situation of Figure 34. Let $\mathcal{X}_\gamma^\pm = \lim_{\vartheta \to \vartheta_c^\pm} \mathcal{X}_\gamma$. Then, one has:

$$\mathcal{X}_\mu^+ = \mathcal{X}_\mu^- (1 - \mathcal{X}_\gamma)^{\langle \mu, \gamma \rangle} \tag{4.29}$$

where γ is the cycle shown in Figure 34, and $\langle , \rangle$ denotes the intersection pairing in $H_1(\Sigma^\circ, \mathbb{Z})$.

Proof. It is sufficient to check this for μ running over a basis of $H_1(\Sigma^\circ, \mathbb{Z})$ modulo the loops around branch points. We have such a basis, corresponding to the internal edges of the triangulation [prove!] and for this basis the $\mathcal{X}_\mu$ are explicit cross-ratios (Proposition 4.29).

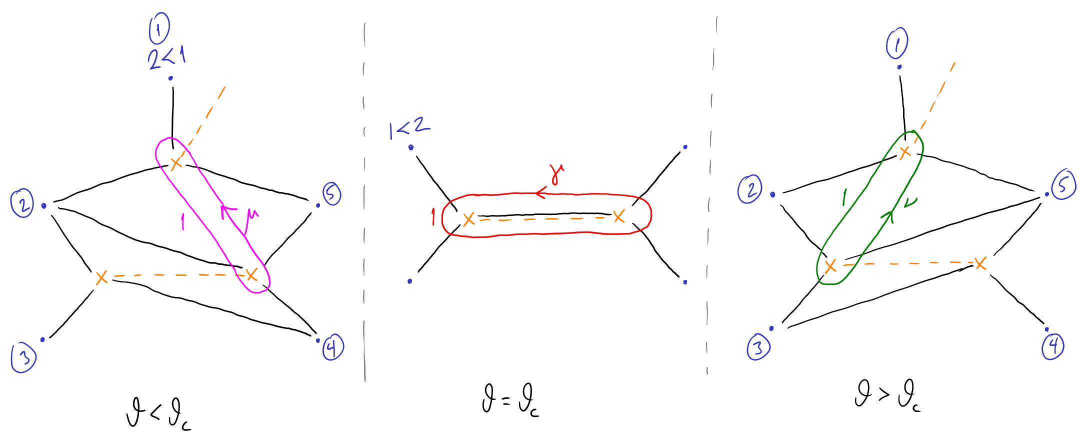

FIGURE 35. Some cycles in a neighborhood of a saddle connection, for computing the induced transformation of the $\mathcal{X}_\mu$.

For example, consider the situation in Figure 35. We have $\langle \mu, \gamma \rangle = -1$. We compute directly using Proposition 4.29

$$\mathcal{X}_\gamma = r(\ell_2, \ell_3, \ell_4, \ell_5), \quad \mathcal{X}_\mu^- = r(\ell_5, \ell_1, \ell_2, \ell_4), \quad \mathcal{X}_\mu^+ = -\mathcal{X}_{\nu-\gamma}^+ = -r(\ell_5, \ell_1, \ell_2, \ell_3)\mathcal{X}_\gamma^{-1} \tag{4.30}$$

These cross-ratios indeed obey the relation

$$\mathcal{X}_\mu^+ = \mathcal{X}_\mu^- (1 - \mathcal{X}_\gamma)^{-1} \tag{4.31}$$

matching (4.29) as desired. By similar computations for the $\mathcal{X}_\mu$ associated to the other edges of the triangulation, we establish (4.29) in full. (It is described in slightly more detail in [25].) □

One can think of the $\mathcal{X}_\gamma$ as coordinates on the relevant moduli space $\mathcal{M}_{\mathrm{Betti}}$. These coordinates are very special and have been discovered in various different contexts. In particular they are part of the structure of "cluster variety" on $\mathcal{M}_{\mathrm{Betti}}$ introduced in [26]. From this point of view the transformation (4.29) is an example of a cluster transformation.
[refs: Voros, KS, DDP, ...]

4.1.10. *Asymptotics of abelianization.* The general philosophy now is that the cluster coordinates $\mathcal{X}_\gamma(\hbar)$ give "good" coordinates in which to study the point $[L_\hbar] \in \mathcal{M}_{\mathrm{Betti}}$. Namely, these particular coordinates have simple asymptotics as $\hbar \to 0$, and the full $\mathcal{X}_\gamma(\hbar)$ can be recovered from those asymptotics, e.g. by Borel summation.

Proposition 4.32 (Uniform asymptotics of $\log \mathcal{X}_\gamma(\hbar)$). Let λ_n be the 1-forms on Σ° appearing in the formal WKB series (4.10). Suppose $\vartheta = \arg \hbar$ is a non-BPS phase, and $\gamma \in H_1(\Sigma^\circ, \mathbb{Z})$. Then in some sector $\mathbb{S}_I$ with $\vartheta \in I$ we have

$$\mathcal{X}_\gamma(\hbar) = \exp \xi_\gamma(\hbar) \tag{4.32}$$

where

$$\xi_\gamma(\hbar) \sim \hbar^{-1} \sum_{n=0}^{\infty} \hbar^n \oint_\gamma \lambda_n \text{ as } \hbar \to 0\,. \tag{4.33}$$

Proof. Let I be a sufficiently small interval that none of the topology below changes as ϑ varies within I.

The intersections $z_1, \ldots, z_k$ between γ and the walls of $\mathcal{W}(\vartheta)$ divide γ into arcs $\gamma_1, \ldots, \gamma_k$. Along the arc γ_k, which runs from z_k to z_{k+1}, we have two bases (i.e. nonzero elements) of $A_\hbar$, given by

$$\psi_k(z) = \exp \hbar^{-1} \int_{z_k}^{z} \lambda, \qquad \psi_k' = \exp \hbar^{-1} \int_{z_{k+1}}^{z} \lambda\,. \tag{4.34}$$

The parallel transport along γ_k maps

$$\psi_k \mapsto \exp\left(\hbar^{-1} \int_{\gamma_k} \lambda\right) \psi_k'\,. \tag{4.35}$$

Now, recall that ψ_k' and ψ_{k+1} are not generally equal, because the form λ can jump at the wall. The parallel transport of $A_\hbar$ involves the gluing map:

$$\psi_k' \mapsto \frac{\psi_k' \wedge \tilde{\psi}_{k+1}}{\psi_{k+1} \wedge \tilde{\psi}_{k+1}} \psi_{k+1} \tag{4.36}$$

where $\tilde{\psi}_{k+1}$ is a solution from the other sheet,

$$\tilde{\psi}_{k+1}(z) = \exp\left(\hbar^{-1} \int_{z_{k+1}}^{z} \tilde{\lambda}\right)\,. \tag{4.37}$$

The gluing factor in (4.36) can be computed using the concrete realization of $\psi \wedge \tilde{\psi}$ as the Wronskian: this gives it as

$$\frac{\lambda - \tilde{\lambda}}{\lambda' - \tilde{\lambda}}(z_{k+1}) \tag{4.38}$$

where $\lambda(z_{k+1})$ and $\lambda'(z_{k+1})$ are two (lateral) Borel summations of the same series $\lambda_{\mathrm{formal}}(z_{k+1}, \hbar)$, and thus they have the same asymptotic series in $\hbar$. It follows that this factor can be written as $\exp a_k(\hbar)$ where $a_k(\hbar) \sim 0$ as $\hbar \to 0$ in $\mathbb{S}_I$. Now putting together the whole parallel transport around the loop γ, we have $\mathcal{X}_\gamma(\hbar) = \exp \xi_\gamma(\hbar)$ where

$$\xi_\gamma(\hbar) = \sum_{k=1}^{m} \hbar^{-1} \int_{\gamma_k} \lambda(\hbar) + a_k, \tag{4.39}$$

and using the asymptotic series expansion $\lambda(\hbar) \sim \lambda_{\mathrm{formal}}(\hbar)$ on each arc, we get the desired asymptotic series (4.33). □

In particular, we can get the leading asymptotic of $\mathcal{X}_\gamma(\hbar)$ by taking the first two terms in $\lambda(\hbar)$ (all other terms are suppressed as $\hbar \to 0$). The first term is λ_0, while the second is $-\hbar \frac{1}{4} \frac{P'}{P}\, \mathrm{d}z$. Integrating this second term gives $\frac{\mathrm{i}}{2}\pi w$, where w is the winding number of $P(z)$ around the origin as we traverse γ. Since w is always even, we can write $\exp(\frac{\mathrm{i}}{2}\pi w) = (-1)^{w/2}$, and summarize as:

Proposition 4.33 (Leading asymptotics of $\mathcal{X}_\gamma(\hbar)$). As $\hbar \to 0$, if $\arg \hbar$ is not a BPS phase, we have

$$\mathcal{X}_\gamma(\hbar) \sim (-1)^{w/2} \exp(\hbar^{-1} Z_\gamma)\,. \tag{4.40}$$

Now let us reconsider the jump (4.29) of $\mathcal{X}_\gamma$. Because of the asymptotics (4.40), this jump is invisible in the $\hbar \to 0$ expansion along the ray $Z_\gamma/\hbar < 0$ — as it must be, since the asymptotic series of $\mathcal{X}_\mu$ holds uniformly, with no jumps. Thus this jump can be understood as an instance of Stokes phenomenon.

4.1.11. *The overall picture.* What we have found is: the Stokes data for the local systems $L_\hbar$ (associated to the Schrödinger equations (4.2)) is conveniently captured by functions $\mathcal{X}_\gamma(\hbar)$ corresponding to cycles $\gamma \in H_1(\Sigma^\circ, \mathbb{Z})$. The functions $\mathcal{X}_\gamma(\hbar)$ have uniform asymptotics as $\hbar \to 0$, given by the WKB series (4.33), and suffer Stokes phenomena at BPS phases $\arg \hbar$, with the Stokes jumps given by (4.29). Thus, the $\mathcal{X}_\gamma(\hbar)$ have a nonuniform meaning: they represent different cross-ratio coordinates of $L_\hbar$ for different values of $\arg \hbar$.

Example 4.34 (The functions $\mathcal{X}_\gamma(\hbar)$ for the Schrödinger equation with cubic potential). For example, let us reconsider the Schrödinger equation (4.2) with cubic potential, $P(z) = z^3 - 1$. In this case there are saddle connections with phases $\vartheta = \frac{\pi}{6} + \frac{n\pi}{3}$. It follows that the $X_\gamma(\hbar)$ suffer Stokes phenomena at 6 rays in the $\hbar$-plane, with the jump in each case given by (4.29). See Figure 36.

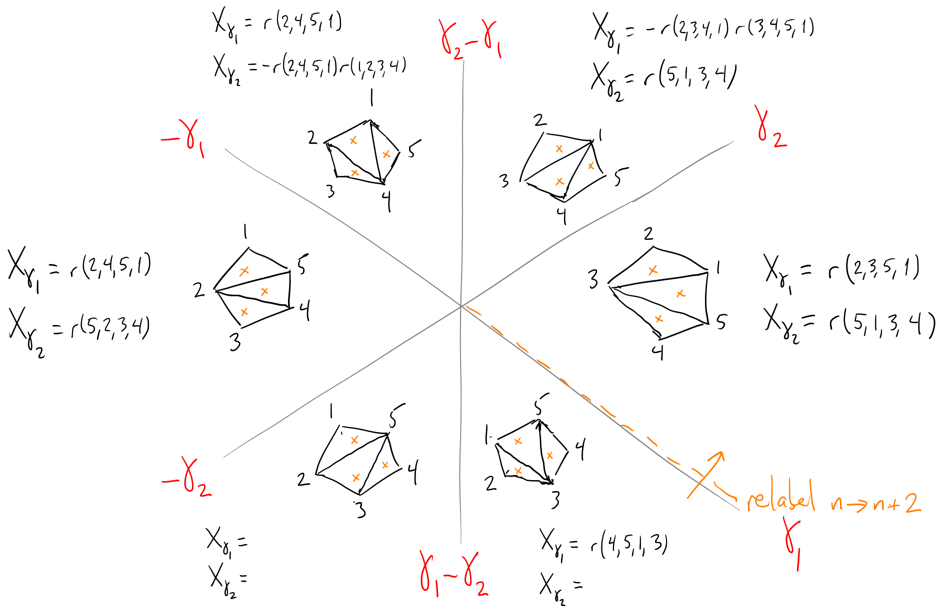

FIGURE 36. The analytic structure of the functions $\mathcal{X}_\gamma(\hbar)$, in the case of the Schrödinger equation with cubic potential $P(z) = z^3 - 1$. In each sector we indicate a labeled triangulation of the pentagon, and the corresponding functions $\mathcal{X}_{\gamma_1}$, $\mathcal{X}_{\gamma_2}$ in terms of cross-ratios of the lines $\ell_1, \ldots, \ell_5$. In going from one sector to the next, the pentagon rotates by an angle $\frac{2\pi}{15}$, i.e. one-third of a unit. Each BPS phase ϑ is labeled by the charge of the corresponding saddle connection γ, with $\arg(-Z_\gamma) = \vartheta$.

Exercise 4.14. Fill in the three missing cross-ratio formulas in Figure 36.

Since each γ_i encircles two branch points, the leading asymptotic (4.40) is

$$\mathcal{X}_{\gamma_i} \sim -\exp(Z_{\gamma_i}/\hbar)\,, \tag{4.41}$$

where the explicit periods are

$$Z_{\gamma_1} = \mathrm{e}^{5\pi\mathrm{i}/6}M, \qquad Z_{\gamma_2} = \mathrm{e}^{-5\pi\mathrm{i}/6}M, \qquad M = \sqrt{3\pi}\frac{\Gamma(4/3)}{\Gamma(11/6)}\,. \tag{4.42}$$

Exercise 4.15. Verify (4.42) (say, using Mathematica).

To actually compute the functions $\mathcal{X}_\gamma$, there are at least two approaches. [refs: Iwaki-Nakanishi, Allegretti, Nikolaev...?]

Conjecture 4.3 (Borel summation computes $\mathcal{X}_\gamma$). The formal series (4.33) is Borel summable whenever $\arg\hbar$ is not a BPS phase, and its Borel summation gives $\xi_\gamma(\hbar)$.

Again, although I call this a conjecture, it is close to a theorem: e.g. [27] contains this statement, with the extra hypothesis that $|\hbar|$ is sufficiently small.

[second conjectural way to get $\mathcal{X}$'s: TBA integral equation]

The upshot of all this is that the local system $L_\hbar$ which we want to understand is completely determined up to isomorphism by the data of the spectral curve Σ, the (classical) periods Z_γ, and the integers $\Omega(\gamma)$ counting the saddle connections.

4.1.12. *Reconstruction.* We have described how to use WKB to build a local system $A_\hbar$ over Σ which is constructed from $L_\hbar$ in a slightly subtle way: $A_\hbar$ describes the WKB gradings of $L_\hbar$ away from the Stokes curves, and is glued together across the Stokes curves using the Stokes automorphisms. In this section we explain how to reconstruct $L_\hbar$ completely from $A_\hbar$. [formalize as an appropriate functor?]

Definition 4.35 (Paths). Fixing $z, z' \in C$ we let $\mathrm{Path}(z, z')$ be the space of paths from z to z' up to homotopy.

Given a path $\mathcal{P} \in \mathrm{Path}(z, z')$ and any local system L over C, we have the parallel transport

$$\mathbb{T}_L(\mathcal{P}) : L_z \to L_{z'} \,. \tag{4.43}$$

Definition 4.36 (Lifted paths). Fixing $z, z' \in C$ we consider formal finite sums

$$Y = \sum_a n_a X_a \tag{4.44}$$

where a runs over homotopy classes of paths on Σ° which begin at preimages of z and end at preimages of z', and all $n_a \in \mathbb{Z}$. Let $\mathrm{LPath}(z, z')$ be the abelian group of such sums, modulo the relation that if a, a' differ by homotopy across a branch point, then $X_a = -X_{a'}$.

We introduce a product structure $\mathrm{LPath}(z, z') \otimes \mathrm{LPath}(z', z'') \to \mathrm{LPath}(z, z'')$ by concatenation: i.e. if a and b can be concatenated to $a + b$ we write $X_a X_b = X_{a+b}$, and if they cannot be concatenated we write $X_a X_b = 0$.

Definition 4.37 (Transport along lifted paths). Suppose given a local system A over Σ. Given an element $X_a \in \mathrm{LPath}(z, z')$ from i to i', and a local system A over Σ, we have the parallel transport $\mathbb{T}_A(a) : A_i \to A_{i'}$, which we can extend by zero to a map

$$\mathbb{T}_A(X_a) : \bigoplus_{i \in \pi^{-1}(z)} A_i \to \bigoplus_{i' \in \pi^{-1}(z')} A_{i'} \,. \tag{4.45}$$

For general $Y = \sum_a n_a X_a \in \mathrm{LPath}(z, z')$ we define

$$\mathbb{T}_A(Y) : \bigoplus_{i \in \pi^{-1}(z)} A_i \to \bigoplus_{i' \in \pi^{-1}(z')} A_{i'} \tag{4.46}$$

by

$$\mathbb{T}_A(Y) = \sum_a n_a \mathbb{T}_A(X_a) \,. \tag{4.47}$$

Now, suppose that we have A and L which are related by abelianization: i.e. we have a Stokes graded local system L over (C, ϕ_2) of some phase ϑ, and its abelianization A over Σ (in the sense of Construction 4.27). In this case, we can recover L from A, as follows. Let $\mathcal{W} = \mathcal{W}(\vartheta)$. For any $z \notin \mathcal{W}$, the Stokes grading gives an isomorphism

$$\iota_z : \bigoplus_{i \in \pi^{-1}(z)} A_i \simeq L_z. \tag{4.48}$$

Construction 4.38 (Path lifting). There is a map

$$F : \mathrm{Path}(z, z') \to \mathrm{LPath}(z, z') \tag{4.49}$$

such that, whenever A and L are related by abelianization,

$$\mathbb{T}_L(\mathcal{P}) = \iota_{z'} \circ \mathbb{T}_A(F(\mathcal{P})) \circ \iota_z^{-1} \,. \tag{4.50}$$

Proof. We begin with the case of $\mathcal{P} \in \mathrm{Path}(z, z')$ which does not intersect $\mathcal{W}$. In this case we define

$$F(\mathcal{P}) = \sum_i \mathcal{P}^{(i)} \tag{4.51}$$

where $\mathcal{P}^{(i)}$ are the lifts to the two sheets of Σ. This F obeys (4.50).

Next suppose $\mathcal{P}$ crosses $\mathcal{W}$ exactly once, as indicated in Figure 37.

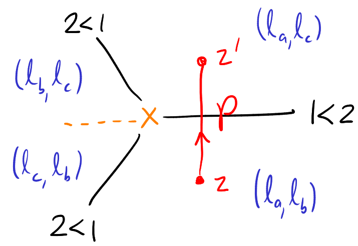

FIGURE 37. A path $\mathcal{P}$ crossing $\mathcal{W}$ once. We also label the line decompositions of L in a neighborhood of the branch point.

In this case we define

$$F(\mathcal{P}) = X_{a_1} + X_{a_2} + X_{a_3} \tag{4.52}$$

where a_1, a_2, a_3 are the paths in Figure 38.

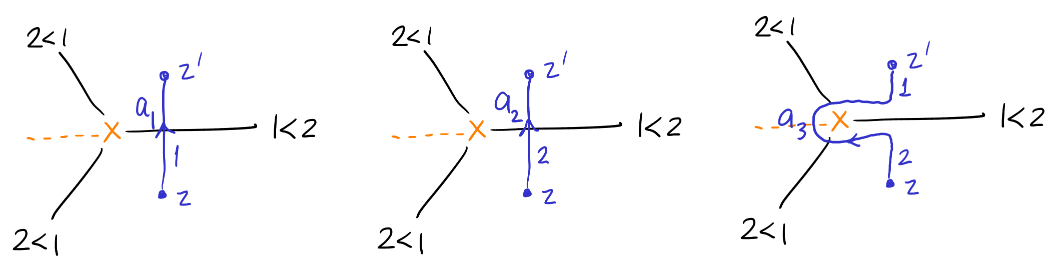

FIGURE 38. Three lifts of $\mathcal{P}$.

To see that this obeys (4.50) we compute directly in the vector space $L(U)$. First, $\iota_{z'} \circ \mathbb{T}_A X_{a_1} \circ \iota_z^{-1}$ kills ψ_b and maps $\psi_a \mapsto \psi_a$. Second, $\iota_{z'} \circ \mathbb{T}_A (X_{a_2} + X_{a_3}) \circ \iota_z^{-1}$ kills ψ_a and maps

$$\psi_b \mapsto \frac{\psi_b \wedge \psi_a}{\psi_c \wedge \psi_a} \psi_c + \frac{\psi_b \wedge \psi_c}{\psi_a \wedge \psi_c} \psi_a = \psi_b \,. \tag{4.53}$$

Thus $\iota_{z'} \circ \mathbb{T}_A F(\mathcal{P}) \circ \iota_z^{-1}$ acts as the identity in $L(U)$, matching $\mathbb{T}_L \mathcal{P}$ as needed.

Finally, for general $\mathcal{P}$, we write $\mathcal{P} = \mathcal{P}_1 \cdots \mathcal{P}_k$ where each $\mathcal{P}_n$ is one of the above two types, and then set $F(\mathcal{P}) = F(\mathcal{P}_1) \cdots F(\mathcal{P}_k)$. Since both sides of (4.50) are multiplicative, and (4.50) holds for each $\mathcal{P}_n$, it also holds for $\mathcal{P}$ as desired. □

An equivalent description of F is as follows: we take the sum of all possible lifts of $\mathcal{P}$ to Σ, where whenever we cross $\mathcal{W}$ we are allowed to splice in the soliton at the wall.

[show an example]

[remark on homotopy invariance]

4.1.13. *Jumps of abelianization.* Now consider the situation in Figure 39.

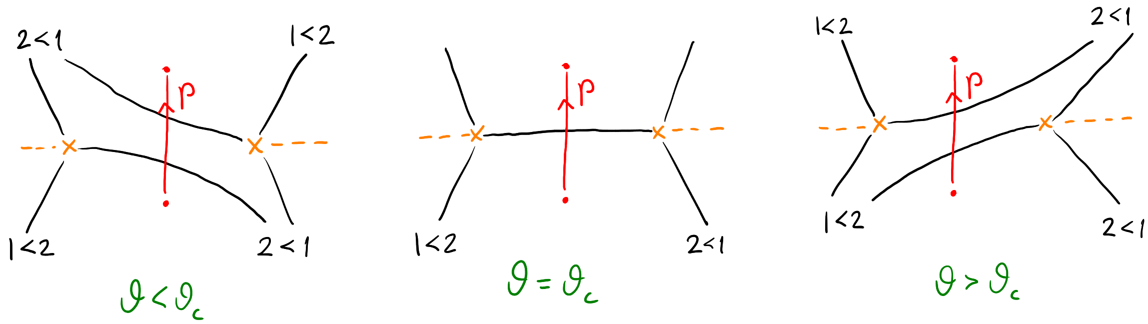

FIGURE 39. Passing through a BPS phase.

At $\vartheta = \vartheta_c$, the critical graph $\mathcal{W}(\vartheta)$ contains a saddle connection. Because of the topology change as ϑ crosses ϑ_c, we have two *a priori* different abelianizations $A^\pm$ of L, obtained by taking limits as $\vartheta \to \vartheta_c^\pm$.

For $Y \in \mathrm{LPath}(z, z')$ let $Y_{ii'}$ be the component consisting of paths from i to i'.

FIGURE 40. The lifted paths appearing in $F^\pm(\mathcal{P})_{11}$.

Then we have

$$F^+(\mathcal{P})_{11} = X_a, \qquad F^-(\mathcal{P})_{11} = X_a + X_b, \tag{4.54}$$

with the paths shown in Figure 40. On the other hand, from (4.50) it follows that

$$\mathbb{T}_{A^+}(F^+(\mathcal{P})) = \mathbb{T}_{A^-}(F^-(\mathcal{P}))\,. \tag{4.55}$$

This means that A has to jump in a way that compensates the jump of F. This gives an alternative way of understanding and proving the formula (4.29).

[develop this further here?] [diagrammatic rule for expanding a given cross ratio?]

4.1.14. *Schrödinger equations with rational potential.* So far we discussed only Schrödinger equations with polynomial potential $P(z)$. These equations have an irregular singularity at $z = \infty$ in $C = \mathbb{CP}^1$. More generally we could consider a rational potential $P(z)$.

In particular, suppose we take a $P(z)$ for which all poles have order ≤ 2 and $\lim_{z\to\infty} zP(z) = 0$. In this case the Schrödinger equation has a regular singularity at each pole of $P(z)$, and possibly also a regular singularity at $z = \infty$. Equivalently (see next section), $\phi_2 = P(z)\,\mathrm{d}z^2$ has poles of order ≤ 2, including possibly at $z = \infty$.

Exercise 4.16. Prove that, if $\lim_{z\to\infty} z^4 P(z)$ exists and is nonzero, then ϕ_2 has neither a pole nor a zero at $z = \infty$.

The simplest situation arises if all poles of ϕ_2 have order exactly 2. In this case, the story of the WKB method is closely analogous to what we have already discussed. For a generic phase ϑ, all critical trajectories emerge from zeroes and end up at poles. If ϕ_2 has only simple zeroes, then the topology of the critical graph $\mathcal{W}(\vartheta)$ is captured as before by a triangulation of C, with vertices at the poles.[31]

Moreover, we can apply Construction 4.27 and Construction 4.38 to study the local system of solutions, just as we did in the polynomial case; all the proofs and constructions apply directly in this case as well.

Here is one thing we did not discuss in the polynomial case (because it would have been trivial there).

Corollary 4.39 (Closed paths on Σ compute traces). Suppose L and A are related by abelianization, $\mathcal{P}$ is a closed path on C, and $F(\mathcal{P}) = \sum_a n_a X_a$. Then

$$\operatorname{Tr} \mathbb{T}_L(\mathcal{P}) = \sum_a n_a \mathbb{T}_A(X_a) \tag{4.56}$$

where the sum runs only over the closed paths a, so that each $\mathbb{T}_A(X_a)$ is a map from a line to itself, i.e. an element of $\mathbb{C}^\times$.

Proof. This follows directly from the formula (4.50) which gives $\mathbb{T}_L(\mathcal{P})$ itself. □

Example 4.40 (Schrödinger equation with 4 regular singularities). The first interesting case arises when the number of poles is 4. The corresponding Schrödinger equation is sometimes called the Heun equation. To be very concrete let us consider the case

$$P(z) = -\frac{z^4 + 2z^2 - 1}{(z^4 - 1)^2}\,. \tag{4.57}$$

With this choice, $\phi_2 = P(z)\,\mathrm{d}z^2$ has four simple zeroes and four poles of order 2, with all residues equal to $-\frac{1}{8}$. The resulting Stokes graph $\mathcal{W}(\vartheta = 0)$ is shown in Figure 41.

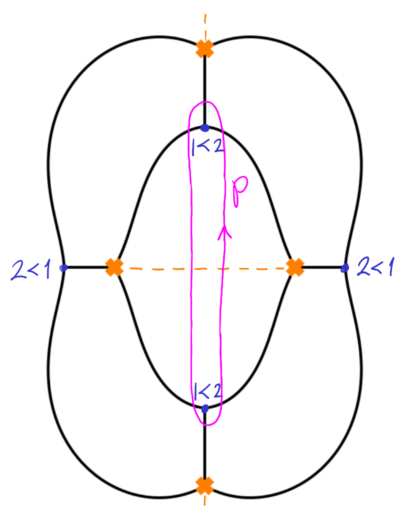

FIGURE 41. The Stokes graph $\mathcal{W}(\vartheta = 0)$ on $\mathbb{CP}^1$ corresponding to the potential (4.57), with a path $\mathcal{P}$ on $\mathbb{CP}^1$ marked.

[31] More precisely, in general we may encounter a *self-folded* triangulation, in which some faces are degenerate. See e.g. [26] for the definition and [25] for some examples where self-folded triangulations arise from quadratic differentials ϕ_2.

Let L be the local system of solutions, and A its abelianization, constructed by Construction 4.27 just as in the polynomial case. Now consider the path $\mathcal{P}$ shown in Figure 41. We want to understand the corresponding trace, $\operatorname{Tr} \mathbb{T}_L(\mathcal{P})$. According to Corollary 4.39 we can compute it as a sum of holonomies over all the closed paths in $F(\mathcal{P})$.

Exercise 4.17. Use the rules of Construction 4.38, applied to Figure 41, to compute $F(\mathcal{P})$. (Hint: there are 11 possible lifted paths, but 4 cancel in pairs, leaving behind 7.)

Exercise 4.18. Use your favorite computation tool to compute (analytically or numerically) the periods Z_γ corresponding to the 7 cycles γ which appear in the result of Exercise 4.17. Use the result and Corollary 4.39 to determine the leading asymptotic behavior of $\operatorname{Tr} \mathbb{T}_L(\mathcal{P})$ as $\hbar \to 0$ with $\arg \hbar = 0$.

Exercise 4.19. Use your favorite computation tool to compute the trace $\operatorname{Tr} \mathbb{T}_L(\mathcal{P})$ numerically for small $\hbar$, and thus verify the result of Exercise 4.18.

Now, as before, we can consider the Stokes phenomena experienced by the functions $\mathcal{X}_\gamma(\hbar)$. At phases ϑ where an isolated saddle connection connecting two zeroes of ϕ_2 appears, the $\mathcal{X}_\gamma(\hbar)$ jump by the formula (4.29), as before. There is also a new phenomenon which did not occur in the polynomial case: there may appear an annulus foliated by closed trajectories.

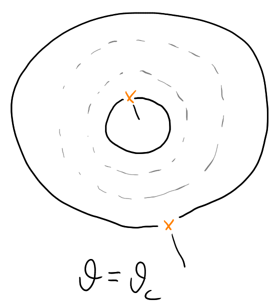

FIGURE 42. An annulus foliated by closed trajectories, with a single branch point on each boundary component.

In this case we again have a Stokes phenomenon, given by a close analog of (4.29):

Proposition 4.41 (Stokes factor associated to a juggle). Consider the situation of Figure 42, in which an annulus appears in $\mathcal{W}(\vartheta)$ at the phase $\vartheta = \vartheta_c$. Let $\mathcal{X}_\mu^\pm = \lim_{\vartheta \to \vartheta_c^\pm} \mathcal{X}_\mu$. Then, one has:

$$\mathcal{X}_\mu^+ = \mathcal{X}_\mu^- (1 + \mathcal{X}_\gamma)^{-2\langle \mu, \gamma \rangle} \tag{4.58}$$

where γ is the cycle shown in [fig].

Proof. See [25] for the computation. One interesting sidelight is that the coordinates $\mathcal{X}_\mu^\pm$ are in a certain sense analogues of complexified Fenchel-Nielsen coordinates. In particular, for the γ shown in [fig], the corresponding $\mathcal{X}_\gamma^+ = \mathcal{X}_\gamma^-$ is one of the eigenvalues of monodromy around the annulus (like the exponential of the Fenchel-Nielsen "length" coordinate). □

[also discuss pop]
[give an example with a ring domain]

4.1.15. *Allowing first-order poles.* [saddle connections initiated at first-order poles]

4.1.16. SL(2)-*opers.* We have been discussing Schrödinger equations on $C = \mathbb{CP}^1$ in order to be concrete. But a similar picture holds more generally for Schrödinger equations on a compact Riemann surface C with meromorphic potential.

The main difficulty in generalizing is to define what we mean by a Schrödinger equation in a coordinate-independent way. We can understand the issue directly by considering a coordinate transformation $z(\tilde{z})$, and letting

$$\alpha(z) = z'(\tilde{z}), \quad \tilde{\psi}(\tilde{z}) = \alpha(z)^a \psi(z(\tilde{z})), \quad \tilde{P}(\tilde{z}) = \alpha(z)^b P(z(\tilde{z}))\,. \tag{4.59}$$

Then we find directly

$$\left(\partial_{\tilde{z}}^2 + \tilde{P}(\tilde{z})\tilde{\psi}(\tilde{z})\right) = \alpha^{-2+a}\left(\alpha^4\psi'' + \alpha^{2+b}P\psi + (1+2a)\alpha^2\alpha'\psi' + (a^2-a)\alpha'^2\psi + a\alpha\alpha''\psi\right) \tag{4.60}$$

In order to eliminate the term involving ψ' on the right side we need $a = -\frac{1}{2}$, and to arrange that the powers of α are the same in the $P\psi$ and ψ'' terms we need $b = 2$. To make sense of the half-integer powers of α we make a choice of $\sqrt{\alpha}$. Then, after making these choices the right side becomes

$$\alpha^{\frac{3}{2}}\left(\psi'' + P\psi\right) + \alpha^{-\frac{5}{2}}\left(\frac{3}{4}\alpha'^2 - \frac{1}{2}\alpha\alpha''\right)\psi \tag{4.61}$$

The second part of this expression looks complicated, but it is proportional to the Schwarzian derivative of $z(\tilde{z})$, and in particular it vanishes if $z(\tilde{z})$ is a Möbius transformation.

This calculation motivates the following definitions:

Definition 4.42 ($\mathbb{CP}^1$ structure). A $\mathbb{CP}^1$ *structure* on C is an atlas of charts where all transition functions are Möbius transformations.

Definition 4.43 (Spin structure). A *spin structure* on C is a line bundle $K_C^{1/2}$ over C, equipped with an isomorphism $K_C^{1/2} \otimes K_C^{1/2} \simeq K_C = T^*C$.

Once we have chosen a spin structure on C, by tensoring and taking duals we can define line bundles $K_C^{n/2}$ for any n. The datum of a spin structure on C will be used below to determine square roots of transition maps $\alpha = z'(\tilde{z})$.

Construction 4.44 (Meromorphic SL(2)-opers). Suppose given both a $\mathbb{CP}^1$ structure and a spin structure on C. Also fix a meromorphic quadratic differential ϕ_2 on C. Then there exists a second-order differential operator

$$\mathcal{D}_\hbar : K_C^{-1/2} \to K_C^{3/2} \tag{4.62}$$

characterized as follows. Fix a patch $U \subset C$ with a local coordinate z in the atlas given by the $\mathbb{CP}^1$ structure. Choose an element $\sqrt{\mathrm{d}z} \in K_C^{1/2}(U)$, with $\sqrt{\mathrm{d}z}^2 = \mathrm{d}z$. Also write $\phi_2 = P(z)\,\mathrm{d}z$ on U. Then

$$\mathcal{D}_\hbar(\psi(z)\,\mathrm{d}z^{-1/2}) = (\hbar^2\partial_z^2 + P(z))\psi(z)\,\mathrm{d}z^{3/2}\,. \tag{4.63}$$

In particular, we see that in order to make the Schrödinger equation coordinate-invariant it needs to act, not on functions $\psi(z)$, but on sections $\psi(z)(\mathrm{d}z)^{-1/2}$.

Everything we have said about the WKB analysis of Schrödinger equations with rational potential generalizes to the case of meromorphic SL(2)-opers on a compact Riemann surface C.[32]

4.1.17. *Donaldson-Thomas invariants.* We can formalize the structure we have found a bit.

Definition 4.45 (Potential walls of marginal stability). We say ϕ_2 lies on a potential wall of marginal stability if there exist $\gamma, \gamma' \in H_1(\Sigma, \mathbb{Z})$ such that $\arg Z_\gamma = \arg Z_{\gamma'}$ but γ, γ' are linearly independent.

The potential walls of marginal stability are the places where the jumping rays for $\mathcal{X}_\gamma$ can collide.

Definition 4.46 (BPS invariants for $N = 2$). Fix C and ϕ_2 as above. Assume that ϕ_2 has no simple poles. Also assume ϕ_2 does not lie on a potential wall of marginal stability. Suppose $\gamma \in H_1(\Sigma, \mathbb{Z})$. Then define

$$\Omega(\gamma) = \#\{\text{saddle connections with charge } \gamma\} - 2\#\{\text{ring domains with charge } \gamma\} \in \mathbb{Z}\,. \tag{4.64}$$

The $\Omega(\gamma)$ are piecewise locally constant functions of ϕ_2; they can jump when ϕ_2 crosses a wall of marginal stability.

The weights 1 and -2 here reflect the way that the saddle connections and ring domains induce Stokes phenomena: see (4.29) and (4.58).

The quantity $\Omega(\gamma)$ is finite, irrespective of any conditions on ϕ_2. In very simple examples, $\Omega(\gamma)$ can even be written explicitly. For instance:

Example 4.47 (BPS invariants for the cubic potential). In our running example of $P(z) = z^3 - 1$, using the basis shown in Figure 32, we have

$$\Omega(\gamma) = \begin{cases} 1 & \text{for } \gamma \in \{\pm\gamma_1, \pm\gamma_2, \pm(\gamma_1 - \gamma_2)\}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{4.65}$$

These six nonzero invariants correspond to the six saddle connections.

Although we cannot develop all the background here, we want to remark that the quantity $\Omega(\gamma)$ defined in this way fits into a broader story in algebraic geometry:

Theorem 4.48 (BPS invariants are DT invariants). (Roughly — see [28] for the precise theorem.) Fix a topological surface S with marked points p_i, labeled by integers n_i, with at least one $n_i \geq 2$. Then there is a triangulated 3-Calabi-Yau category $\mathcal{C}$ such that:

- for each (complex structure on S and meromorphic ϕ_2, with a pole of order n_i at p_i for each i), there is a corresponding Bridgeland stability condition on $\mathcal{C}$,
- the BPS invariants $\Omega(\gamma)$ are the generalized Donaldson-Thomas invariants associated to this category and stability condition.

Very recently it was shown that the same theorem is also true without the marked points. [Haiden]

[32] In particular, the construction of the formal WKB series λ_{formal} generalizes. To understand this carefully we need to understand the meaning of the last term $\hbar\partial_z\lambda_z$ in the equation (4.9). The key fact is that given a $\mathbb{CP}^1$ structure on C and a 1-form λ on Σ there is a canonically defined quadratic differential on C whose local coordinate expression is $\partial_z\lambda_z\, dz^2$.

4.2. **WKB for higher rank opers.** Now we consider equations of rank $N \geq 2$. We will not repeat the gradual development we did for Schrödinger equations: rather we begin with opers on a general surface immediately. All of the structures we considered for $N = 2$ have upgrades to $N \geq 2$ which we now sketch.

4.2.1. *Tuples of differentials.* We will consider tuples

$$\vec{\phi} = (\phi_2, \ldots, \phi_N) \tag{4.66}$$

where ϕ_k is a meromorphic section of K_C^k, i.e. a meromorphic k-differential.

Definition 4.49 (Spectral curve). Given such a tuple we have the spectral curve

$$\Sigma = \left\{ \sum_{i=0}^{N} \phi_i \, s^{N-i} = 0 \right\} \subset T^*C\,, \tag{4.67}$$

where we define $\phi_0 = 1$.

When Σ is smooth and reduced, the projection $\pi : \Sigma \to C$ is a branched N-fold cover.

Example 4.50 (Spectral curve for $N = 3$). For $N = 3$ the spectral curve specializes to

$$\Sigma = \left\{ s^3 + \phi_2 s + \phi_3 = 0 \right\} \subset T^*C\,. \tag{4.68}$$

See Figure 43 for a cartoon.

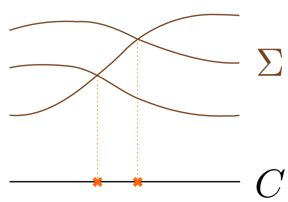

FIGURE 43. The spectral curve Σ in the case $N = 3$.

Example 4.51 (A simple $N = 3$ spectral curve). Suppose we take $C = \mathbb{CP}^1$, and $\vec{\phi}$ given by

$$\phi_2 = -3\mathrm{d}z^2, \qquad \phi_3 = 2z\,\mathrm{d}z^3\,. \tag{4.69}$$

Then the spectral curve is concretely (using $s = y\,\mathrm{d}z$)

$$\Sigma = \{y^3 - 3y + 2z = 0\} \subset \mathbb{C}^2\,. \tag{4.70}$$

The branch locus is determined by the vanishing of the discriminant of this cubic equation: $\Delta = 108(1 - z^2)$, so there are branch points at $z = \pm 1$. Moreover, since these are simple zeroes of Δ it follows that the ramification around each is simple, involving only 2 of the 3 sheets. See Figure 44.

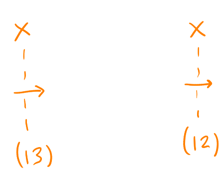

FIGURE 44. A representation of the 3-fold cover $\Sigma \to C$, when $\vec{\phi}$ is given by (4.69). The orange crosses represent the two branch points, at $z = \pm 1$.

Exercise 4.20. Prove that Σ is smooth.

Exercise 4.21. Use the Riemann-Hurwitz formula and the structure of the branching to show that Σ is a once-punctured sphere.

Because $H_1(\Sigma, \mathbb{Z})$ is trivial we will not have any interesting abelian holonomies $\mathcal{X}_\gamma$ in this case.

A little more generally we could take $\phi_2 = -3\epsilon\, \mathrm{d}z^2$, and then the branch points will be at $z = \pm\epsilon^{3/2}$. In the limit $\epsilon \to 0$, the two branch points collide, giving a single branch point, which is a double root of the discriminant Δ. Nevertheless, the spectral curve Σ is still smooth when $\epsilon = 0$.

4.2.2. *Opers.*

Construction 4.52 ($\mathrm{SL}(N)$**-opers).** Suppose given a $\mathbb{CP}^1$ structure and (if N is even) a spin structure on C. Also fix a meromorphic tuple $\vec{\phi}$ on C. Then there exists an N-th order differential operator

$$\mathcal{D}_\hbar : K_C^{-(N-1)/2} \to K_C^{(N+1)/2} \tag{4.71}$$

characterized as follows. Fix a patch $U \subset C$ with a local coordinate z in the atlas given by the $\mathbb{CP}^1$ structure. Choose an element $\sqrt{\mathrm{d}z} \in K_C^{1/2}(U)$, with $\sqrt{\mathrm{d}z}^2 = \mathrm{d}z$. Also write $\phi_k = P_k(z)\, \mathrm{d}z$ on U. Then

$$\mathcal{D}_\hbar(\psi(z)\, \mathrm{d}z^{-(N-1)/2}) = (O_\hbar \psi(z))\, \mathrm{d}z^{(N+1)/2} \tag{4.72}$$

where $O_\hbar$ is the differential operator obtained by replacing $s \mapsto \hbar\partial_z$ and $\phi_k \mapsto P_k$ in the equation (4.67), with a specific resolution of the ordering ambiguity:

$$Ps \rightsquigarrow \hbar\left(\frac{1}{2}P\partial_z + \frac{1}{2}\partial_z P\right), \tag{4.73}$$

$$Ps^2 \rightsquigarrow \hbar^2\left(\frac{3}{10}P\partial_z^2 + \frac{2}{5}\partial_z P\partial_z + \frac{3}{10}\partial_z^2 P\right), \tag{4.74}$$

and similar formulas for general Ps^k.

Proof. We just need to check that the operator defined in this way is independent of the choice of local coordinate, i.e. we need to check that our definition of $O_\hbar$ is covariant under Möbius transformations. For translation $z \mapsto z + a$ and scaling scaling $z \mapsto az$ this is straightforward, and does not depend on the choice of ordering prescription. Requiring covariance under $z \mapsto z^{-1}$ on the other hand is sufficient to determine the ordering prescription. □

Another description of this construction (for general N but with less explicit formulas) can be found in [29] or [30] (and references therein).

Example 4.53 ($\mathrm{SL}(3)$**-opers).** If $N = 3$, the differential operator $O_\hbar$ can be written concretely as

$$O_\hbar\psi = \hbar^3\partial_z^3\psi + \frac{1}{2}\hbar(P_2\partial_z\psi + \partial_z(P_2\psi)) + P_3\psi \tag{4.75}$$

or equivalently

$$O_\hbar = \hbar^3 \partial_z^3 + \hbar P_2 \partial_z + \left(\frac{1}{2} \hbar \partial_z P_2 + P_3 \right) . \tag{4.76}$$

Exercise 4.22. Verify directly that the differential operator $\mathcal{D}_\hbar$ given in Example 4.53 is well defined, i.e. that it is independent of the choice of coordinate in the atlas.

We can rewrite the equation $\mathcal{D}_\hbar \psi = 0$ as $\nabla_\hbar \psi = 0$ for an appropriate meromorphic $\mathrm{SL}(N)$-connection $(\mathcal{F}_\hbar, \nabla_\hbar)$ over C. [explain this in more detail] When the differentials $\vec{\phi}$ have poles, the connection $\nabla_\hbar$ has poles. If, for some n, ϕ_n has a pole of order $> n$, then the corresponding singularity of $\nabla_\hbar$ is an irregular singularity (see Exercise C.1). The N asymptotic exponents are obtained by integrating λ_0 over the N sheets of Σ, and the covering Θ by asymptotic exponents can be identified with Σ near each irregular singularity. [expand this a bit, prove it]

4.2.3. *The WKB series.* We fix a local coordinate z on $U \subset C$, in the atlas given by the $\mathbb{CP}^1$ structure. We want to study local solutions of

$$O_\hbar \psi = 0. \tag{4.77}$$

As we did before, we write a formal solution in the form

$$\psi_{\text{formal}}(z) = \exp \left(\hbar^{-1} \int_{z_0}^{z} \lambda_{\text{formal}} \right) \tag{4.78}$$

Requiring that this solves (4.77) leads to a Riccati-type equation for λ_{formal}; for $N = 2$ this equation was (4.9); for $N = 3$ it is

$$\lambda^3 + P_2 \lambda + P_3 + \hbar \left(\frac{1}{2} \partial_z P_2 + 3 \lambda \partial_z \lambda \right) + \hbar^2 \partial_z^2 \lambda = 0. \tag{4.79}$$

Again as before, the Riccati equation has one formal solution for each sheet i of Σ, with an expansion of the form

$$\lambda^{(i)}_{\text{formal}} = \sum_{n=0}^{\infty} \hbar^n \lambda^{(i)}_n \tag{4.80}$$

or more explicitly (suppressing the index i)

$$\lambda_{\text{formal}} = \lambda_0 + \hbar \frac{-P_2 P_2' + 6 \lambda_0 P_3' + 3 \lambda_0^2 P_2'}{2(P_2 + 3\lambda_0^2)^2} + \cdots \tag{4.81}$$

which in the special case $P_2 = 0$ becomes

$$\lambda_{\text{formal}} = \lambda_0 - \hbar \frac{P_3'}{3 P_3} + \cdots \tag{4.82}$$

4.2.4. *Foliations.* For equations of higher rank we will meet a richer spectrum of Stokes phenomena. WKB gives in this case N local solutions $\psi^{(i)}$. There are Stokes curves where $\psi^{(i)}$ jumps by a multiple of $\psi^{(j)}$ with all other $\psi^{(k)}$ acting as bystanders. Thus we will be interested in keeping track systematically not only of individual sheets i of Σ but also of pairs (i, j). This motivates the following:

Definition 4.54 (Root curve). Suppose Σ is smooth. The *root curve* is the closure of the fiber product with diagonal removed,

$$\widetilde{\Sigma} = \overline{\{(s,s') \in \Sigma \times \Sigma : \pi(s) = \pi(s'), s \neq s'\}} \subset T^*C \times T^*C. \tag{4.83}$$

This is a smooth $(N^2 - N)$-sheeted branched covering of C, perhaps with some points deleted over the poles of $\vec{\phi}$.

There is a natural involution of $\widetilde{\Sigma}$ given by $\mu(s,s') = (s',s)$. Projection on the first and second factors gives maps $p_1, p_2 : \widetilde{\Sigma} \to \Sigma$, with $p_2 = p_1 \circ \mu$.

Example 4.55 (Root curve for $N = 2$). For $N = 2$, the root curve is just another language for talking about Σ and its standard covering involution: indeed the maps p_1, p_2 are both isomorphisms $\widetilde{\Sigma} \simeq \Sigma$, differing by composition with the covering involution. Under either of these isomorphisms, moreover, μ is taken to the covering involution.

Example 4.56 (Root curve for $N = 3$). For $N = 3$, the situation is already more interesting: $\widetilde{\Sigma}$ is a 6-fold covering of C, while Σ is a 3-fold covering, and the maps p_1, p_2 are branched double covers. Above a simple branch point, $\widetilde{\Sigma}$ has 3 ramification points, all of index 1 (i.e. the 6 sheets come together in 3 pairs). But these 3 ramification points are different: one is fixed by the involution μ while the other two are exchanged.

If we trivialize Σ locally then there is an induced local trivialization of $\widetilde{\Sigma}$, labeling its sheets by ordered pairs (i,j) with $1 \leq i,j \leq N$ and $i \neq j$. So paths on $\widetilde{\Sigma}$ can be represented by pictures like Figure 45.

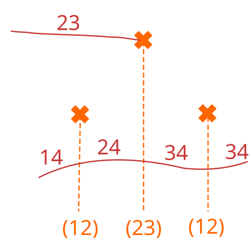

FIGURE 45. A representation of paths on $\widetilde{\Sigma}$ by paths on the base C, labeled by ordered pairs of sheets of Σ.

Definition 4.57 (Canonical 1-form on the root curve). Let λ_0 denote the canonical (Liouville) 1-form on T^*C; its restriction is a holomorphic 1-form on Σ. Define

$$\rho = p_1^* \lambda_0 - p_2^* \lambda_0. \tag{4.84}$$

ρ is a holomorphic 1-form on $\widetilde{\Sigma}$, with zeroes only over the branch points.

Using this 1-form we can define the foliation of $\widetilde{\Sigma}$ by ϑ-trajectories, just as we did before:

Definition 4.58 (Action coordinate). Given a simply connected domain $U \subset \widetilde{\Sigma}$ we consider a map $w : U \to \mathbb{C}$ obeying the condition

$$\mathrm{d}w = \rho. \tag{4.85}$$

If ρ does not vanish on U, then w gives a coordinate on U (possibly after shrinking U). This coordinate is canonical up to a constant shift $w \to w + c$.

Definition 4.59 (ϑ-trajectories on $\widetilde{\Sigma}$). A ϑ-trajectory on $\widetilde{\Sigma}$ is a path along which $\operatorname{Im} \mathrm{e}^{-\mathrm{i}\vartheta} w$ is constant. It is canonically oriented in the direction in which $\operatorname{Re} \mathrm{e}^{-\mathrm{i}\vartheta} w$ decreases. Let F^ϑ denote the foliation of $\widetilde{\Sigma}$ by ϑ-trajectories.

We have $\mu^* \rho = -\rho$. It follows that μ preserves F^ϑ but reverses its orientation.

Example 4.60 (Foliations for $N = 3$). Around a generic point the projection of F^ϑ to C looks (topologically, not conformally!) like:

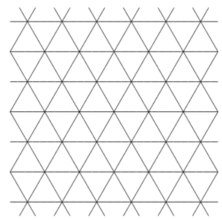

FIGURE 46. A topological picture of the projection of F^ϑ to C, in a neighborhood where Σ is trivializable.

There are also some degenerate loci (caustics) where the three foliations become parallel. Around a branch point it is hard to draw the full picture, but we can draw at least part of the projection: it has the same three-pronged structure that we know from the $N = 2$ case.

4.2.5. *Solitons.* If c is a 1-chain on $\widetilde{\Sigma}$, let $p(c)$ be a 1-chain on Σ given by

$$p(c) = (p_1)_* c - (p_2)_* c \,. \tag{4.86}$$

(So if in a local trivialization c carries the label ij then $p(c)$ consists of two components, one carrying label i and one carrying label j, with the second one oppositely oriented.) Note it has

$$\int_{p(c)} \lambda = \int_c \rho. \tag{4.87}$$

Definition 4.61 (Topological solitons). Fix $z \in C$ and $s, s' \in \pi^{-1}(z)$. A *topological soliton from s to s'* is a 1-chain c on $\widetilde{\Sigma}$, such that $\partial(p(c)) = s' - s$.

The projection of a topological soliton from $\widetilde{\Sigma}$ to C looks like a graph (generically trivalent), with one leaf at z and all other leaves at branch points. The labels on the edges of the graph are constrained: there is a balancing condition at each internal vertex, and also a condition at each branch point, coming from the condition that $\partial(p(c))$ doesn't contain any points lying over the internal vertices or branch points. Examples of projections of solitons obeying the conditions are shown below.

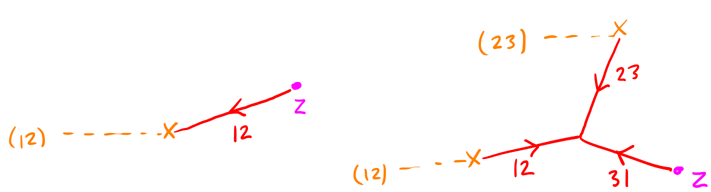

FIGURE 47. Topological solitons.

Definition 4.62 (WKB solitons). Fix $z \in C$ and $s, s' \in \pi^{-1}(z) \subset \Sigma$. A *WKB soliton from s to s' with phase ϑ* is a topological soliton from s to s' made up of paths which are all ϑ-trajectories.

So the projection of a WKB soliton to C is a graph as for a topological soliton, but now with some rigidity: the edges are required to be leaves of the foliation with matching labels.

Example 4.63 (A multi-leg WKB soliton). We consider the $\vec{\phi}$ given in Example 4.51. In this case, for some values of $z \in C$, there exist multi-leg WKB solitons ending at z; an example is indicated in Figure 48.

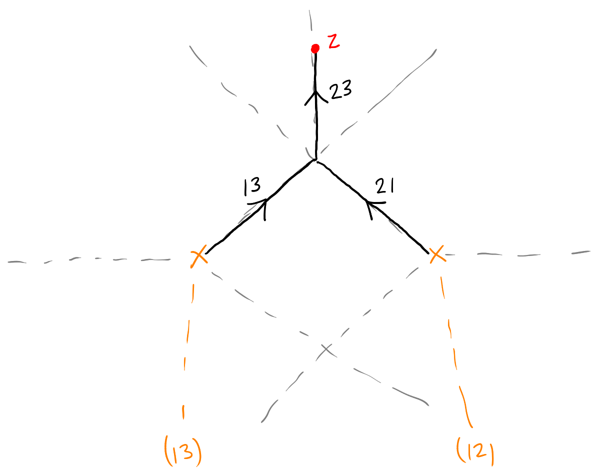

FIGURE 48. A multi-leg WKB soliton in Example 4.51.

The main conjecture is that the WKB solitons here play the same role that they played in the $N = 2$ case:

Conjecture 4.4 (Borel summability of the WKB series for $\mathrm{SL}(N)$-opers). When $\lambda_{\mathrm{formal},z}$ is the WKB series associated to an $\mathrm{SL}(N)$-oper, the Borel transform $\mathcal{B}\lambda_{\mathrm{formal},z}(\zeta)$ at $s \in \Sigma$ admits analytic continuation to the largest star-shaped region around 0 which does not contain any periods of WKB solitons beginning at s. This continuation obeys an exponential bound $|\mathcal{B}\lambda_{\mathrm{formal},z}(\zeta)| < \mathrm{e}^{M|\zeta|}$ for every $M > 0$.

From now on we assume Conjecture 4.4. It would be very interesting to establish it.

It follows from Conjecture 4.4 that by Borel summation we can construct local solutions $\psi^{(i)}(\hbar, z)$ which are piecewise holomorphic functions of $z \in C$. The solution $\psi^{(i)}$ can jump at those $z \in C$ for which there exists a WKB soliton with phase $\vartheta = \arg \hbar$ from sheet i to some sheet j over z. A sharper version of this statement is that $\psi^{(i)}$ can jump only by a multiple of the solution $\psi^{(j)}$:

Conjecture 4.5 (Stokes condition on jumps of WKB solutions for $\mathrm{SL}(N)$-opers). Suppose $z \in C$ and there exists a WKB soliton over z with phase ϑ, beginning on sheet i and ending on sheet j. Let $\psi^{(i)}_\pm = \lim_{\arg \hbar \to \vartheta^\pm} \psi^{(i)}$. Then $\psi^{(i)}_+ - \psi^{(i)}_- = c\psi^{(j)}$ for some $c \in \mathbb{C}$.

[add many refs]

4.2.6. *Spectral network.*

Definition 4.64 (WKB spectral network). The *WKB spectral network* $\mathcal{W}(\vartheta)$ is the set of all points $(s, s') \in \widetilde{\Sigma}$ such that there exists a WKB soliton from s to s' with phase ϑ.

Example 4.65 (WKB spectral network when $N = 2$). When $N = 2$, all WKB solitons are segments whose projection runs from z to a branch point. It follows that the projection of $\mathcal{W}(\vartheta)$ to C is the critical graph, agreeing with what we discussed above.

Proposition 4.66 (Algorithm for computing $\mathcal{W}(\vartheta)$). If Σ is smooth and all ramification points are simple, then we can compute $\mathcal{W}(\vartheta)$ as follows. For each branch point p, there is a unique μ-fixed ramification point $r_p \in \widetilde{\Sigma}$ lying over p. ρ has a second-order zero at r_p, so there are three leaves of F^ϑ passing through r_p as indicated in Figure 19. Let $\mathcal{C}$ be the smallest set of *half-leaves* of F^ϑ such that:

- $\mathcal{C}$ must include, for each p, the three half-leaves beginning at r_p which are oriented away from r_p.

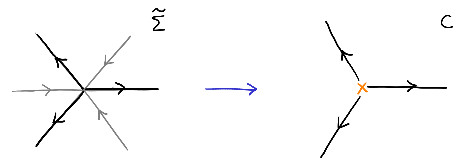

FIGURE 49. Three half-leaves which are included in $\mathcal{C}$.

- Suppose $\ell_1, \ell_2 \in \mathcal{C}$ are such that $p_1(\ell_1)$ intersects $p_2(\ell_2)$ transversely at a point $s \in \Sigma$, with preimages $(s, s_1) \in \ell_1$ and $(s_2, s) \in \ell_2$. Then $\mathcal{C}$ must also include a third half-leaf beginning at (s_2, s_1).

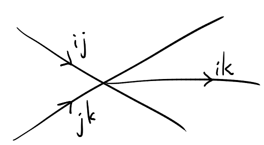

FIGURE 50. The projection of these three half-leaves to the base C, in a neighborhood of $\pi(s)$.

We emphasize that the intersection point $\pi(s) \in C$ is a regular point, neither a branch point of $\Sigma \to C$ nor a singularity of $\vec{\phi}$; the only thing that is special about this point is that it happens to be the place where $p_1(\ell_1)$ and $p_2(\ell_2)$ intersect.
- (This one is technical: in the generic case it will not be needed.) Suppose $\ell \in \mathcal{C}$ is a half-leaf which runs into a singularity of F^ϑ at a point $r \in \Sigma$, which is not μ-fixed. (In our local notation on C, this means a leaf labeled ij running into a branch point of type jk, for $k \neq i$.) Then ρ has a first-order zero at r, so the foliation F_ϑ has a 4-fold point at r. In this case $\mathcal{C}$ must include both of the half-leaves beginning at r which are oriented away from r.

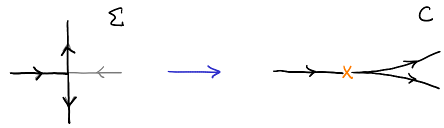

FIGURE 51. The non-generic situation in which a trajectory of type ij accidentally runs into a branch point of type jk.

Then $\mathcal{W}(\vartheta)$ is the union of all half-leaves in $\mathcal{C}$.

A half-leaf in $\mathcal{W}(\vartheta)$ which lies on sheet ji of $\widetilde{\Sigma}$ corresponds to a WKB soliton running from sheet i to sheet j on Σ. The corresponding Stokes phenomena involve $\psi^{(j)}$ jumping by a multiple of $\psi^{(i)}$. Thus walls labeled ji will correspond to orderings $j \prec i$.
[describe construction of $A_\hbar$]

4.2.7. *Some examples.* We consider some examples with $N = 3$, on $C = \mathbb{CP}^1$, with $\phi_2 = P_2 \mathrm{d}z^2$, $\phi_3 = P_3 \mathrm{d}z^3$.

Example 4.67 (A simple spectral network with $N = 3$). In the case of Example 4.51, where $P_2 = -3$ and $P_3 = 2z$, the spectral network $\mathcal{W}(\vartheta = \frac{\pi}{2})$ is shown in Figure 52.

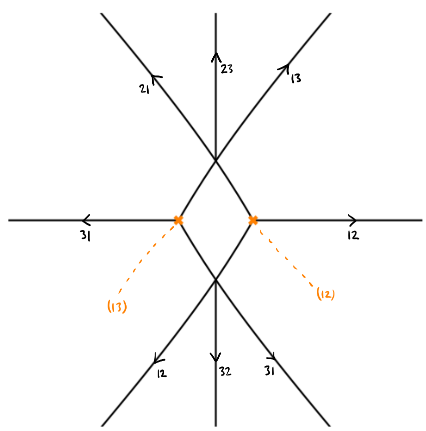

FIGURE 52. The spectral network $\mathcal{W}(\vartheta = \frac{\pi}{2})$ for $P_2 = -3$, $P_3 = 2z$.

If we take $P_2 = -3\epsilon$ and then let $\epsilon \to 0$, this picture degenerates: the two branch points coalesce into a single branch point at $z = 0$, with non-simple ramification, but Σ is nevertheless smooth. The spectral network $\mathcal{W}(\vartheta)$ in the degeneration limit is given by Figure 53. In particular note that in this limit there is no distinction between the 8 trajectories: all of them emanate directly from the branch point.

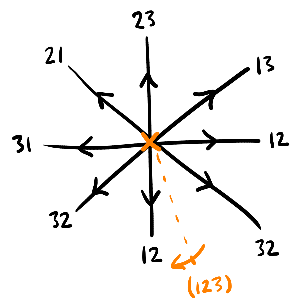

FIGURE 53. The spectral network $\mathcal{W}(\vartheta = \frac{\pi}{2})$ for $P_2 = 0$, $P_3 = 2z$.

Exercise 4.23. Derive Figure 53 directly from the definition of $\mathcal{W}(\vartheta)$ in terms of WKB solitons, rather than from a limiting process.

Example 4.68 (A single triangle with $N = 3$). In case $P_2 = z$ and $P_3 = -1$ the network $\mathcal{W}(\vartheta = 0)$ is shown in Figure 54.

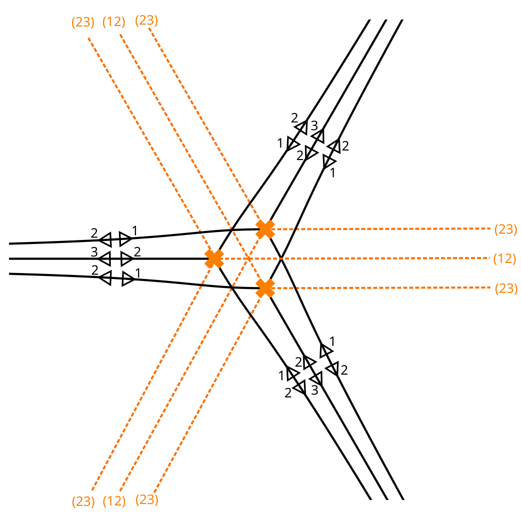

FIGURE 54. The spectral network $\mathcal{W}(\vartheta = 0)$ corresponding to $P_2 = z$, $P_3 = -1$.

The corresponding SL(3)-oper is

$$\left(\hbar^3 \partial_z^3 + \hbar z \partial_z - 1\right) \psi(z) = 0\,. \tag{4.88}$$

This equation has an irregular singularity at $z = \infty$, with asymptotic exponents $\Theta = \left(\frac{2}{3}(-z)^{\frac{3}{2}}, 0, -\frac{2}{3}(-z)^{-\frac{3}{2}}\right)$. On the boundary circle ∂, there are three Stokes lines with labels $3 \prec 2 \prec 1$, separated by three branch cuts carrying the permutation (13). This is an instance of Exercise 3.8; as shown there, the corresponding moduli space $\mathcal{M}_{\mathrm{Betti}}$ of Stokes graded local systems is the configuration space of triples of complete flags in $\mathbb{C}^3$.

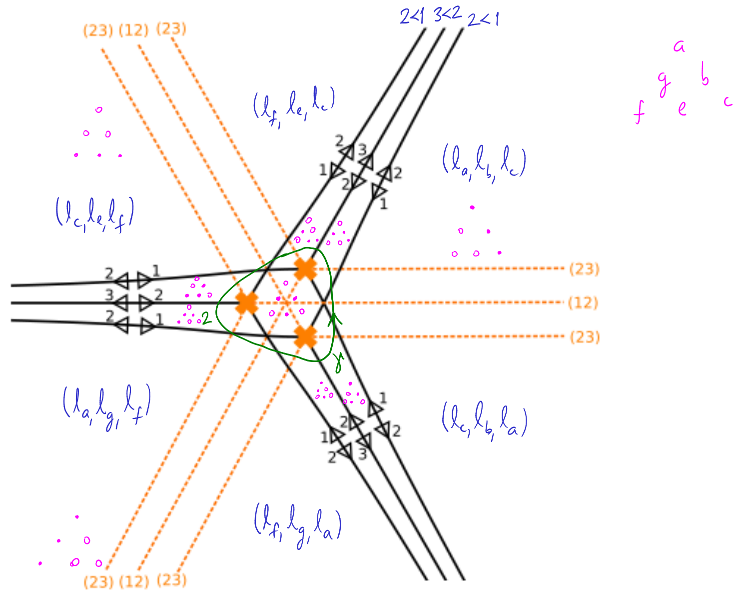

FIGURE 55. The spectral network as above, decorated by the data of the Stokes gradings in each domain. At upper left we indicate how the six lines can be conveniently arranged in a triangle. In the large domains we list the three lines explicitly; in the small domains we just mark solid dots on a triangle to indicate which three lines are used in the grading.

Now we consider the Stokes gradings provided by WKB. See Figure 55. The data of a point of $\mathcal{M}_{\text{Betti}}$ corresponds to giving the Stokes gradings in the three large domains around $z = \infty$ in the figure. These gradings involve six lines in the 3-dimensional vector space V of global solutions. These six lines are conveniently arranged in a triangle as shown in Figure 55. They are subject to constraints following from the Stokes conditions at the walls: for each upward-pointing subtriangle, the three lines at the vertices are coplanar inside V. So e.g. (ℓ_a, ℓ_b, ℓ_g) are coplanar. The WKB method provides Stokes gradings not only in these three big regions but in *all* the regions of the figure.

Exercise 4.24. Show that the Stokes gradings in the remaining regions are as indicated in the figure. In particular, show that they are completely determined by the gradings in the big regions around $z = \infty$.

Exercise 4.25. Check that the spectral curve Σ in this case is a twice-punctured sphere, so $H_1(\Sigma, \mathbb{Z})$ has rank 1.

Now we consider the cycle $\gamma \in H_1(\Sigma, \mathbb{Z})$ shown in Figure 55. The computation of the holonomy of $A_\hbar$ around γ goes along similar lines to Proposition 4.29. Since γ lies on sheet 2, the nontrivial gluing factors will arise only from the three walls labeled $3 \prec 2$. These factors can be computed using the result of Exercise 3.11. Writing (a, b, c) for $\psi_a \wedge \psi_b \wedge \psi_c$, combining these three factors gives

$$\mathcal{X}_\gamma = \frac{(b,c,g)}{(e,c,g)} \frac{(e,b,f)}{(g,b,f)} \frac{(g,a,e)}{(b,a,e)} . \tag{4.89}$$

This quantity is the Fock-Goncharov *triple ratio*, introduced in [26]. It gives a coordinate on $\mathcal{M}_{\mathrm{Betti}}$. Because we have identified it with a holonomy of $A_\hbar$ we get our usual asymptotic consequences,

$$\mathcal{X}_\gamma(\hbar) \sim c \exp(Z_\gamma/\hbar) \tag{4.90}$$

[compute Z_γ as a residue, note this formula is actually exact]

Example 4.69 (Two triangles). In case $P_2 = z^2 - 1$ and $P_3 = \epsilon < 0$ the network $\mathcal{W}(\vartheta = 0)$ is shown in Figure 56. [check phase]

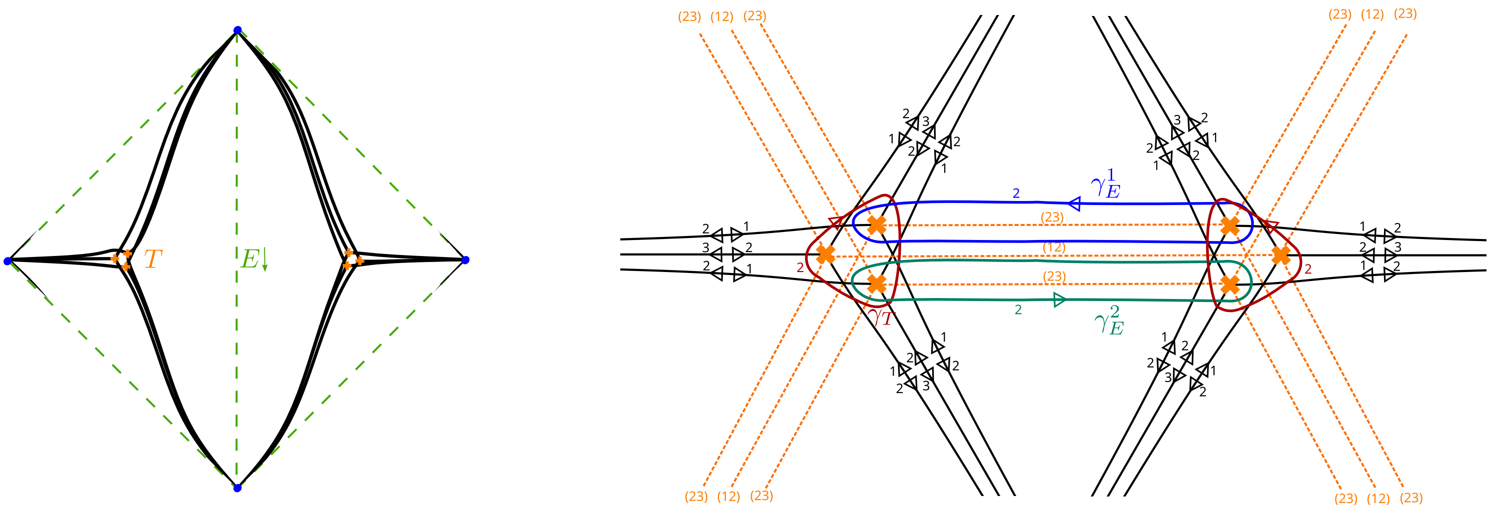

FIGURE 56. Right: the spectral network $\mathcal{W}(\vartheta = 0)$ corresponding to $P_2 = z^2 - 1$, $P_3 = \epsilon < 0$, with four cycles in $H_1(\Sigma, \mathbb{Z})$ marked. Left: a corresponding triangulation of the square.

The four $\mathcal{X}_\gamma$ associated to the four cycles shown above give local coordinates on $\mathcal{M}_{\mathrm{Betti}}$ in this case. [and it's similar for a more general triangulated surface, one coordinate per triangle and two per edge, cf Fock-Goncharov]

Example 4.70 (A polynomial spectral network with $N = 3$). Here is a different example, not directly related to a triangulated surface. In case $P_2 = 0$ and $P_3 = -z^2 + 1$ the network $\mathcal{W}(\vartheta = 0)$ is shown in Figure 57.

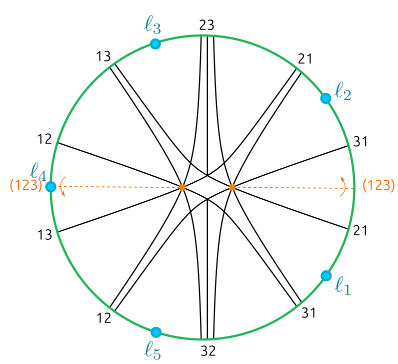

FIGURE 57. The spectral network $\mathcal{W}(\vartheta = 0)$ corresponding to $P_2 = 0$, $P_3 = -z^2 + 1$.

In this case, the relevant Betti space is $\mathcal{M}_{\mathrm{Betti}} = \mathrm{Gr}(3,5)/(\mathbb{C}^\times)^5$, i.e. the space of 3×5 matrices, modulo multiplication by $\mathrm{GL}(3)$ on the left and $(\mathbb{C}^\times)^5$ on the right. [two cross-ratio coordinates]

Example 4.71 (Another polynomial spectral network with $N = 3$). In case $P_2 = \epsilon$ and $P_3 = \frac{1}{2}(-z^3 + 3z^2 + 2)$ the network $\mathcal{W}(\vartheta = 0.1)$ is shown in Figure 58.

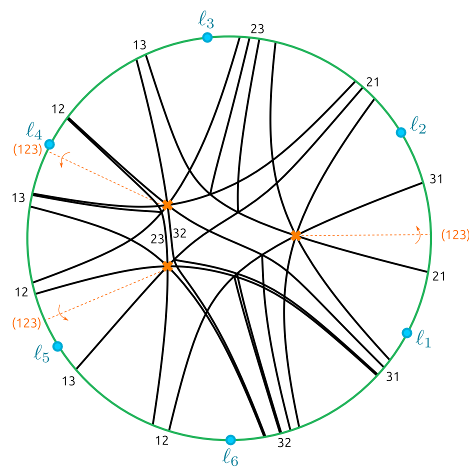

FIGURE 58. The spectral network $\mathcal{W}(\vartheta = 0.1)$ corresponding to $P_2 = 0$, $P_3 = \frac{1}{2}(-z^3 + 3z^2 + 2)$.

[$\mathcal{M}_{\mathrm{Betti}} = \mathrm{Gr}(3,6)$ and we get more interesting coordinates: some Plucker and some non-Plucker]
[if we vary while holding asymptotic picture fixed, we get all the clusters?]

[comment on finiteness question]

4.2.8. *Donaldson-Thomas invariants.*

Definition 4.72 (Topological finite web). A *topological finite web* is a 1-chain c on $\widetilde{\Sigma}$, such that $\partial(p(c)) = 0$.

Definition 4.73 (WKB finite web). A *WKB finite web with phase* ϑ is a topological finite web made up of paths which are all ϑ-trajectories.

In the case $N = 2$ the WKB finite webs are the same as the finite trajectories from [...], i.e. saddle connections and closed loops.
[show pictures]
[comment we get DT invariants much as in $SL(2)$ case]
[show example of a 3-string web]

5. APPLICATIONS TO HIGGS BUNDLES

5.1. **Families of connections.** In [...] we have been considering particular families of differential operators $\mathcal{D}_\hbar$, which degenerate as $\hbar \to 0$. To talk meaningfully about the

$\hbar \to 0$ limit it is useful to phrase things in such a way that the $\hbar \to 0$ limit is actually finite. Roughly the idea is that we are dealing with a family of connections of the form

$$\nabla_\hbar = \hbar^{-1}\varphi + D + \cdots, \tag{5.1}$$

i.e. with a first-order pole at $\hbar = 0$.

To formulate this more precisely:

Definition 5.1 (Meromorphic λ-connection). For any $\lambda \in \mathbb{C}$, a *meromorphic λ-connection* over C with poles only at D is a pair $(\mathcal{F}, \nabla)$, where $\mathcal{F}$ is a locally free sheaf of $\mathcal{O}_C(*D)$-modules, and ∇ is a map of sheaves

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_C(*D)} \Omega^1_C(*D) \tag{5.2}$$

obeying the λ-Leibniz rule

$$\nabla(fs) = \lambda s \otimes \mathrm{d}f + f\nabla s. \tag{5.3}$$

This notion might seem unfamiliar, so let us note:

- For all $\lambda \neq 0$, any λ-connection is of the form $\lambda\nabla$ where ∇ is an ordinary connection.
- When $\lambda = 0$, a λ-connection is equivalent to a meromorphic Higgs bundle:

  Definition 5.2 (Meromorphic Higgs bundle). A *meromorphic Higgs bundle* over C with poles only at D is a pair $(\mathcal{F}, \varphi)$, where $\mathcal{F}$ is a locally free sheaf of $\mathcal{O}_C(*D)$-modules, and φ is a section of $\mathrm{End}(\mathcal{F}) \otimes \Omega^1_C(*D)$.

Now, each $\mathcal{D}_\hbar$ is equivalent to a connection $(\mathcal{E}_\hbar, \nabla_\hbar)$, in such a way that the family of $\hbar$-connections $(\mathcal{E}_\hbar, \hbar\nabla_\hbar)$ can be extended over $\hbar = 0$, with the $\hbar = 0$ fiber a Higgs bundle $(\mathcal{E}, \varphi)$. [should say briefly what a family is?]

Let us see concretely how this works for $N = 2$. In each coordinate chart we consider the object

$$\Psi = \begin{pmatrix} -\hbar^2 \psi_z' \\ \hbar \psi_z \end{pmatrix} \tag{5.4}$$

When we change coordinate charts, say by a map $\tilde{z}(z)$, Ψ transforms by a matrix of the form

$$\tilde{\Psi} = \begin{pmatrix} \alpha^{\frac{1}{2}} & \hbar(\cdots) \\ 0 & \alpha^{-\frac{1}{2}} \end{pmatrix} \Psi \tag{5.5}$$

where $\alpha = \tilde{z}'(z)$. So globally we can think of Ψ as a section of a rank 2 vector bundle $\mathcal{E}_\hbar$, glued together by transition maps of this form.

On the other hand we have

$$\left[\partial_z + \hbar^{-1}\begin{pmatrix} 0 & P_2 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} -\hbar^2 \psi_z' \\ \hbar \psi_z \end{pmatrix} = \begin{pmatrix} -\hbar^2 \psi_z'' + P_2 \psi_z \\ 0 \end{pmatrix}. \tag{5.6}$$

Thus the local system of solutions of $\mathcal{D}_\hbar \psi = 0$ is $\mathrm{Sol}(\mathcal{E}_\hbar, \nabla_\hbar)$, where in local coordinates

$$\nabla_\hbar = \partial_z + \hbar^{-1}\begin{pmatrix} 0 & P_2 \\ 1 & 0 \end{pmatrix}. \tag{5.7}$$

Now we consider taking $\hbar \to 0$. In this limit the transition maps (5.5) degenerate to

$$\tilde{\Psi} = \begin{pmatrix} \alpha^{\frac{1}{2}} & 0 \\ 0 & \alpha^{-\frac{1}{2}} \end{pmatrix} \Psi \tag{5.8}$$

which is the transition map of the bundle $\mathcal{E} = K_C^{\frac{1}{2}} \oplus K_C^{-\frac{1}{2}}$. The connection (5.7) does not have an $\hbar \to 0$ limit, but

$$\hbar \nabla_\hbar = \hbar \partial_z + \begin{pmatrix} 0 & P_2 \\ 1 & 0 \end{pmatrix} \tag{5.9}$$

limits as $\hbar \to 0$ to

$$\varphi = \begin{pmatrix} 0 & P_2 \\ 1 & 0 \end{pmatrix} . \tag{5.10}$$

Thus we have the family $(\mathcal{E}_\hbar, \hbar\nabla_\hbar)$ of $\hbar$-connections limiting to the Higgs bundle $(\mathcal{E}, \varphi)$.

More generally we might consider any family $(\mathcal{E}_\hbar, \hbar\nabla_\hbar)$ which limits to a Higgs bundle in this way. We expect that WKB analysis will work for any such family, along similar lines to what we have discussed so far for families of opers. We discuss some aspects of this below.

5.2. **Geometry of Higgs bundles.** We suppose given a meromorphic Higgs bundle $(\mathcal{E}, \varphi)$ over C.

Definition 5.3 (Spectral curve). The *spectral curve* is the locus

$$\Sigma = \{(z,s) \in T^*C : \det(\varphi(z) - s) = 0\} \subset T^*C . \tag{5.11}$$

[define moduli space of Higgs bundles, Hitchin fibration]
[key theorem: for C compact and φ holomorphic, Hitchin equations admit a solution]
[resulting families of flat connections]
[diffeomorphism from Higgs bundles to flat connections]
[computing using WKB method]
[hyperkahler structure]
...

Appendix A. The method of complex steepest descent

I strongly recommend [15] and [31] as a reference for this material.

Proposition A.1 (Complex steepest descent). Suppose W is a holomorphic function on some neighborhood of a contour $\gamma \subset \mathbb{C}$, and:

- $\operatorname{Im} W$ is constant along γ,
- $\operatorname{Re} W$ has a unique global minimum along γ, at an interior point s_* with $W'(s_*) = 0$ and $W''(s_*) \neq 0$.

Then

$$\int_\gamma \mathrm{e}^{-W/\hbar}\, \mathrm{d}s \sim \sqrt{\frac{2\pi\hbar}{W''(s_*)}} \mathrm{e}^{-W(s_*)/\hbar} \qquad \text{as } \hbar \to 0 \text{ in } \mathbb{R}_+ , \tag{A.1}$$

where the branch of $\sqrt{W''(s_*)}$ is chosen in concert with the orientation of the contour, as follows: write the contour as $\gamma(t)$, oriented in the direction of increasing t, with $\gamma(t_*) = s_*$; then choose the square root so that $\gamma'(t_*)\sqrt{W''(s_*)} > 0$.

The point s_* in Proposition A.1 is called a *saddle point* of the integral, and the contour γ is called a *saddle-point contour*, *steepest descent contour* or *Lefschetz thimble*.

Example A.2 (Gaussian integral). If $W(s) = \frac{1}{2}s^2$ and γ is the real axis, then we have a single saddle point at $s_* = 0$ and the asymptotic formula in Proposition A.1 is exact:

$$\int_{-\infty}^{\infty} \mathrm{e}^{-s^2/2\hbar}\,\mathrm{d}s \;=\; \sqrt{2\pi\hbar}\,. \tag{A.2}$$

Exercise A.1. Prove Proposition A.1. (You can do it by comparing to a Gaussian integral; use the fact that both for the integral we want to study and for the corresponding Gaussian integral, the dominant contribution as $\hbar \to 0$ comes from a small neighborhood of s_*.)

To study integrals $\int_\gamma \mathrm{e}^{-W(s)/\hbar}\,\mathrm{d}s$ over more general contours, one can sometimes use Cauchy's theorem to deform γ to a union of saddle-point contours, and then apply Proposition A.1.

With more effort one can obtain a full asymptotic expansion, not only its leading term.

Appendix B. Some preliminaries

In this section we collect some topological/categorical preliminaries.

B.1. Sheaves.

Suppose X is a topological space, and $\mathcal{D}$ is a category (e.g. sets, abelian groups, vector spaces are good examples).

Definition B.1 (Presheaf). Let $\mathcal{C}(X)$ be the category whose objects are open subsets of X, and whose morphisms are inclusions. A presheaf on X valued in $\mathcal{D}$ is a functor $\mathcal{F} : \mathcal{C}(X)^{\mathrm{op}} \to \mathcal{D}$. Concretely, this means that for every open set $U \subset X$ we have $\mathcal{F}(U) \in \mathrm{Ob}(\mathcal{D})$, and for every pair $V \subset U$ we have a restriction morphism $r_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$, with $r_{VW} \circ r_{UV} = r_{UW}$. We sometimes write $s|_V$ for $r_{UV}(s)$.

Definition B.2 (Morphism of presheaves). A morphism of presheaves, $\rho : \mathcal{F} \to \mathcal{G}$, is a natural transformation $\rho : \mathcal{F} \implies \mathcal{G}$. Concretely, this means that for every open set $U \subset X$ we have a map $\rho_U : \mathcal{F}(U) \to \mathcal{G}(U)$, compatible with the restriction morphisms.

Exercise B.1. Prove that presheaves on X with values in $\mathcal{D}$ form a category.

Definition B.3 (Restriction of presheaves). For $\mathcal{F}$ a presheaf on X, and $Y \subset X$ open, the restriction $\mathcal{F}|_Y$ is a presheaf on Y obtained by composing $\mathcal{F}$ with the inclusion $\mathcal{C}(Y)^{\mathrm{op}} \to \mathcal{C}(X)^{\mathrm{op}}$. (Thus, for $U \subset Y$ open, $\mathcal{F}|_Y(U) = \mathcal{F}(U)$.)

Definition B.4 (Sheaf). A sheaf is a presheaf $\mathcal{F}$ obeying two additional conditions: whenever (U_α) is an open cover of X,

- Suppose $s, s' \in \mathcal{F}(X)$, and $s|_{U_\alpha} = s'|_{U_\alpha}$ for all α. Then $s = s'$.
- Suppose $s_\alpha \in \mathcal{F}(U_\alpha)$ for all α, agreeing on overlaps, i.e. $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ for all α, β. Then there exists some $s \in \mathcal{F}(X)$ such that $s|_{U_\alpha} = s_\alpha$ for all α.

Example B.5 (Sheaf of continuous maps). For any topological space Y, there is a sheaf on X with values in the category of sets, given by $\mathcal{F}(X) = C(X, Y)$.

Example B.6 (Sheaf of smooth maps). If X and Y are smooth manifolds, there is a sheaf on X with values in the category of sets, given by $\mathcal{F}(X) = C^\infty(X, Y)$.

Definition B.7 (Sheaf of modules). If $\mathcal{G}$ is a sheaf over X valued in the category of $\mathbb{C}$-algebras, and $\mathcal{F}$ is a sheaf over X valued in the category of $\mathbb{C}$-vector spaces, we say $\mathcal{F}$ is a *sheaf of $\mathcal{G}$-modules* or just *$\mathcal{G}$-module* if $\mathcal{F}(U)$ is equipped with the structure of a module over $\mathcal{G}(U)$, and $(a \cdot s)|_V = a|_V \cdot s|_V$. A morphism of $\mathcal{G}$-modules is a morphism ρ of sheaves such that each ρ_U is a homomorphism of $\mathcal{G}(U)$-modules.

From now on we assume the category $\mathcal{D}$ is one of the category of sets, abelian groups or $\mathbb{C}$-vector spaces.

Definition B.8 (Stalk). Given a presheaf $\mathcal{F}$ valued in $\mathcal{D}$, and $x \in X$, the stalk $\mathcal{F}_x$ is the direct limit (colimit) of $\mathcal{O}(U)$ over all open $U \subset X$ containing x.

In particular, given $s \in \mathcal{F}(U)$ and $x \in U$ we get an element $s_x \in \mathcal{F}_x$, called the germ of s at x.

Definition B.9 (Sheafification). Given a presheaf $\mathcal{F}$ valued in $\mathcal{D}$, its sheafification $\mathcal{F}^\#$ is a sheaf defined as follows: $\mathcal{F}^\#(U)$ is the set of all $s \in \prod_{x \in U} \mathcal{F}_x$ such that, for any $x \in U$, there exists some $V \subset U$ containing x, and some $t \in \mathcal{F}(V)$, such that $t_y = s_y$ for $y \in V$.

B.2. **Local systems.**

Definition B.10 (Constant presheaf). The constant presheaf on X with value $C \in \mathrm{Ob}(\mathcal{D})$ is given by $\mathcal{F}(U) = C$ for all U, and $r_{UV} = 1$ for all $V \subset U$.

Definition B.11 (Constant sheaf). The constant sheaf on X with value $C \in \mathrm{Ob}(\mathcal{D})$ is the sheafification of the constant presheaf on X with value C. This sheaf is sometimes denoted $\underline{C}$.

Concretely, $\underline{C}(U)$ is the space of maps $U \to C$ which are constant on each connected component of U.

Definition B.12 (Local system). A sheaf $\mathcal{F}$ on X is a local system (locally constant sheaf) if, for all $x \in X$, there exists a neighborhood U of x such that $\mathcal{F}|_U$ is isomorphic to a constant sheaf.

Construction B.13 (Equivalence between local systems and representations). Assume X is connected, locally path-connected and semilocally simply connected. Choose $x_0 \in X$. Then there is an equivalence between the category of local systems on X valued in N-dimensional vector spaces, where for morphisms we take only isomorphisms of local systems, and the category of N-dimensional representations of $\pi_1(X; x_0)$.

Here is how the construction goes, in outline:[33]

- Given a representation ρ of $\pi_1(X; x_0)$ we want to make a local system L over X. Recall that the universal cover $\widetilde{X}$ has $\widetilde{X}/\pi_1(X; x_0) = X$, and let $\pi : \widetilde{X} \to X$ be the projection map. Now we consider the trivial local system $\underline{\mathbb{C}}^N$ over $\widetilde{X}$. We can lift the action of $\pi_1(X)$ from $\widetilde{X}$ to $\underline{\mathbb{C}}^N$, using the representation ρ, and then taking the

[33] A careful account can be found e.g. at `https://www.math.lsu.edu/~pramod/tc/07s-7280/notes3.pdf`.

quotient by $\pi_1(X)$ gives the desired local system L over X. Said otherwise,

$$L(U) = \{\psi \in \underline{\mathbb{C}}^N(\pi^{-1}(U)) : \psi(\gamma x) = \rho(\gamma)\psi(x) \text{ for } \gamma \in \pi_1(X; x_0)\}, \tag{B.1}$$

with the induced restriction maps.

- Given a local system L, we want to make a representation ρ of $\pi_1(X; x_0)$. Continuation along chains of simply connected open sets gives a homomorphism $\pi_1(X; x_0) \to \mathrm{Aut}(L_{x_0})$. Choosing a basis $L_{x_0} \simeq \mathbb{C}^N$ this becomes a representation $\pi_1(X; x_0) \to \mathrm{GL}(N, \mathbb{C})$ as desired.

B.3. **Sheaves over complex manifolds.** Now we assume X is a complex manifold.

Definition B.14 (Structure sheaf). The *structure sheaf* $\mathcal{O}_X$ is a sheaf valued in the category of $\mathbb{C}$-algebras, defined as follows: for any $U \subset X$, $\mathcal{O}_X(U)$ is the set of all holomorphic functions on U; the restriction maps r_{UV} are given by the conventional restriction of functions.

Definition B.15 (Sheaves of meromorphic functions). We define a sheaf $\mathcal{K}_X$ valued in the category of $\mathbb{C}$-algebras, as follows: for any $U \subset X$, $\mathcal{K}_X(U)$ is the set of all meromorphic functions on U; the restriction maps r_{UV} are given by the conventional restriction of functions. Also let $\mathcal{O}_X(*D)$ be the subsheaf of $\mathcal{K}_X$ consisting of functions which are holomorphic away from D.

We will be particularly interested in $\mathcal{O}_X$-modules.

Example B.16 (Free $\mathcal{O}_X$-modules). There is an $\mathcal{O}_X$-module $\mathcal{O}_X^{\oplus N}$ defined by $\mathcal{O}_X^{\oplus N}(U) = \mathcal{O}_X(U)^{\oplus N}$, with the module structure defined componentwise i.e. $f \cdot (f_1, \ldots, f_N) = (ff_1, \ldots, ff_N)$.

Definition B.17 (Locally free $\mathcal{O}_X$-modules). If $\mathcal{F}$ is an $\mathcal{O}_X$-module, we say $\mathcal{F}$ is *locally free of rank* N if there exists an open covering (U_α) of X such that $\mathcal{F}|_{U_\alpha} \simeq \mathcal{O}_X^{\oplus N}|_{U_\alpha}$.

Any locally free $\mathcal{O}_X$-module is the sheaf of sections of some holomorphic vector bundle over X, and conversely. With this in mind we sometimes call a locally free $\mathcal{O}_X$-module a holomorphic vector bundle. Similarly, we refer to a locally free $\mathcal{O}_X(*D)$-module as a meromorphic vector bundle, with poles along D. (This terminology is less standard.)

Definition B.18 (Fiber of a holomorphic vector bundle). If $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module, and $p \in X$, then the fiber $\mathcal{F}_{[p]}$ is the finite-dimensional vector space $\mathcal{F}_p/\mathfrak{m}_p\mathcal{F}_p$ where $\mathfrak{m}_p \subset (\mathcal{O}_X)_p$ is the ideal consisting of functions vanishing at p.

If $\mathcal{F}$ is locally free of rank N, then $\mathcal{F}_{[p]}$ is an N-dimensional vector space.

Definition B.19 (Tensor product of $\mathcal{O}_X$-modules). If $\mathcal{F}$, $\mathcal{G}$ are $\mathcal{O}_X$-modules, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

Definition B.20 (Meromorphic sections). If $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module, and $D \subset X$, let $\mathcal{F}(*D) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D)$.

Definition B.21 (Sheaf of holomorphic vector fields). The sheaf $\mathcal{T}_X$ of holomorphic vector fields on X is an $\mathcal{O}_X$-module defined as follows: $\mathcal{T}_X(U)$ is the $\mathbb{C}$-vector space of derivations of $\mathcal{O}_X(U)$, i.e. operators $V : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ obeying $V(fg) = V(f)g + fV(g)$.

Definition B.22 (Dual sheaf). If $\mathcal{F}$ is an $\mathcal{O}_X$-module, the dual $\mathcal{F}^*$ is an $\mathcal{O}_X$-module defined as follows: $\mathcal{F}^*(U)$ is the space of all morphisms of $\mathcal{O}_X$-modules from $\mathcal{F}|_U$ to $\mathcal{O}_X|_U$.

Definition B.23 (Sheaf of holomorphic 1-forms). The sheaf Ω^1_X of holomorphic 1-forms on X is the dual of $\mathcal{T}_X$.

B.4. **Equivalences and moduli.** Recall that a functor $F : \mathcal{C} \to \mathcal{C}'$ is (one half of) an equivalence of categories if F is full, faithful and essentially surjective. "Full and faithful" means that for any $x, y \in \mathcal{C}$, $F : \mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{C}'}(F(x), F(y))$ is a bijection. "Essentially surjective" means that every $y \in \mathcal{C}'$ is isomorphic to some $F(x)$.

We will mainly consider categories in which every morphism is an isomorphism: such a category is called a *groupoid*.[34]

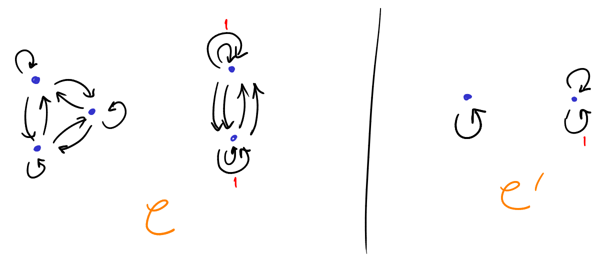

FIGURE 59. Two equivalent categories $\mathcal{C}, \mathcal{C}'$.

Exercise B.2. Construct full, faithful and essentially surjective functors $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}$, where $\mathcal{C}, \mathcal{C}'$ are groupoids as shown in Figure 59. (Note that neither of F and G is uniquely determined; how many possibilities are there for each?)

Given a (small) category $\mathcal{C}$ we can consider the set $\mathcal{M}(\mathcal{C})$ of objects up to isomorphism (invertible morphism). This set is preserved under equivalence of categories, in the appropriate sense:

Exercise B.3. Prove that an equivalence of categories $\mathcal{C} \to \mathcal{C}'$ induces a bijection $\mathcal{M}(\mathcal{C}) \to \mathcal{M}(\mathcal{C}')$.

But an equivalence of categories gives more than just a bijection of moduli sets. For instance, it also identifies the automorphism groups of the objects, namely, if $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence and $F(x) = y$, then F gives an isomorphism $\mathrm{Hom}_{\mathcal{C}}(x,x) \to \mathrm{Hom}_{\mathcal{C}'}(y,y)$. We can see this in the example of Figure 59 above.

B.5. **Real blow-ups.**

Definition B.24 (Real blow-up at a point). If X is a smooth manifold and $p \in X$, the *real blow-up of X at p* is a manifold $\widehat{X}_p$ with boundary, defined as follows. Fix any norm on T_pX and let $\iota : X \to T_pX$ be any smooth map with $\iota(p) = 0$ and $D\iota(p) = 1$. Then $\widehat{X}_p$ is the closure of the image of the map $\varphi : X \setminus \{p\} \to X \times T_pX$ given by $\varphi(x) = (x, \frac{\iota(x)}{\|\iota(x)\|})$.

[34]As partial motivation for the name, note that for any object x in a groupoid $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(x,x)$ is a group; in particular a groupoid with one object is a group.

Exercise B.4. Prove that $\widehat{X}_p$ is indeed a manifold with boundary, and independent of the choices we made.

As a set, $\widehat{X}_p = (X \setminus \{p\}) \sqcup (T_pX \setminus \{0\})/\mathbb{R}_+$. Informally, the real blow-up should be thought of as replacing the point p by the sphere of directions in T_pX. It is homeomorphic to the manifold-with-boundary one obtains by deleting an ϵ-ball around p for small enough ϵ (in any metric), and loosely it can be thought of as the $\epsilon \to 0$ limit of that construction.

Appendix C. Singularities of linear ODE

We consider linear ODE of degree N in one variable:

$$\sum_{n=0}^{N} f_{N-n}(z)\partial_z^n \psi(z) = 0 \tag{C.1}$$

where $f_0(z) = 1$, and all $f_n(z)$ are holomorphic in some punctured neighborhood of $z = 0$.

Definition C.1 (Regular and irregular singularities for scalar ODE in one variable). The singularity of (C.1) at $z = 0$ is *regular* if each $f_n(z)$ has at worst a pole of order n; otherwise the singularity is *irregular*.

Exercise C.1. Explain why an ODE which has a regular singularity at $z = 0$ in this sense corresponds to a connection which has a regular singularity at $z = 0$ in the sense of Definition 2.4.

For singularities at $z = \infty$ there is a similar picture, related by the change of coordinates $z \mapsto 1/z$.

C.1. The Frobenius method.

For an ODE with regular singularity at $z = 0$ one can build solutions by the Frobenius method: one writes an ansatz of the form

$$\psi(z) = z^\lambda \sum_{n=0}^{\infty} a_n z^n \tag{C.2}$$

with $a_0 = 1$. Then (C.1) implies that λ is a root of a characteristic polynomial of degree N. Let's assume this equation has N roots $\lambda_1, \ldots, \lambda_N$, such that no $\lambda_i - \lambda_j \in \mathbb{Z}$. Then choose $\lambda = \lambda_i$ for some i. Plugging (C.2) back into (C.1) one gets a linear recurrence which determines the rest of the a_n. The series thus obtained has a nonzero radius of convergence, so it gives an honest solution ψ_i of (C.1) (in any domain where we choose a branch of z^λ); varying over $i = 1, \ldots, N$ we get a basis for the space of solutions.

In particular, this means that for ODE with regular singularities the solutions have moderate growth, in accord with Proposition 2.13.

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